KEY

Math 70	TUFTS UNIVERSITY	Feb 25, 2013
Linear Algebra	Department of Mathematics	All sections
	Exam 1	

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	10	
3	10	
4	14	
5	12	
6	4	
7	8	
8	16	
9	8	
10	8	
	100	

1. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

   (a) Every homogeneous system of linear equations is consistent. T F

   (b) Every homogeneous system of three linear equations in five unknowns has a nontrivial solution. T F

   (c) Every set of three vectors in $\mathbb{R}^5$ is linearly dependent. T F

   (d) Every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^5$ is one-to-one. T F

   (e) If a collection $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent, then one of the vectors is a scalar multiple of one of the others. T F

2. (10 points)

(a) Write the augmented matrix of the following linear system and row reduce it to echelon form.

$$2x_1 + 10x_2 = -6$$
$$hx_1 + 20x_2 = k$$

$$\left[\begin{array}{cc|c} 2 & 10 & -6 \\ h & 20 & k \end{array}\right] \left(\tfrac{1}{2}\right) \sim \left[\begin{array}{cc|c} 1 & 5 & -3 \\ h & 20 & k \end{array}\right] -hR_1 \sim \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & 20-5h & k+3h \end{array}\right]$$

(b) Find all value(s) of h and k so the linear system above has

(i) no solutions

$20-5h=0 \Rightarrow 5h=20 \Rightarrow h=4$

and

$k+3h \neq 0$ $\quad k+12 \neq 0$ $\quad k \neq -12$

$h=4, k \neq -12$

(ii) only one solution

$20-5h \neq 0 \Rightarrow h \neq 4$

$h \neq 4$

(iii) infinitely many solutions

$20-5h=0 \Rightarrow h=4$

and

$k+3h=0$ $\quad k+12=0$ so $k=-12$

$h=4$

$k=-12$

(c) For which values of h is the associated matrix of coefficients invertible?

Need $2 \cdot 20 - 10h \neq 0$

$40-10h \neq 0$

$40 \neq 10h$

$h \neq 4$

3. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Is A in reduced row-echelon form? If not, give the reduced row-echelon matrix which is row equivalent to A.

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} -R_2 \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Let

$$\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Is the vector equation $A\mathbf{x} = \mathbf{b}$ consistent? If it is, give an explicit solution. If not, explain why not.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] -R_2 \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

parametric vector form

$x_1 + 2x_2 - x_4 = 0$
$x_2 = x_2$
$x_3 + 3x_4 = 2$
$x_4 = x_4$
$x_5 = 1$

$x_1 = -2x_2 + x_4$
$x_2 = x_2$
$x_3 = 2 - 3x_4$
$x_4 = x_4$
$x_5 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

(c) Let the mapping $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be defined by $T(\mathbf{x}) = A\mathbf{x}$. Does T map $\mathbb{R}^5$ *onto* $\mathbb{R}^3$? Justify your answer.

yes - A has a pivot position in every row so $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^3$

4. (14 points) Suppose that the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5]$ is row equivalent to the matrix

$$B = \begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

$x_1 - 2x_3 + 3x_5 = 0$ | $x_1 = 2x_3 - 3x_5$

$x_2 = x_2$ | $x_2 = x_2$

$x_3 = x_3$ | $x_3 = x_3$

$x_4 - 4x_5 = 0$ | $x_4 = 4x_5$

$x_5 = x_5$ | $x_5 = x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

(b) Write the *definition* of what if means for the columns of A to be linearly independent.

$\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$ is LI if the only [soln] to

$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_5\vec{a}_5 = \vec{0}$

is $c_1 = c_2 = \cdots = c_5 = 0$

(c) Are the columns of A linearly independent? If so, justify. If not, find a dependence relation among the columns of A.

Not L.I. There are infinitely many sol'ns to $A\vec{x} = \vec{0}$ also. (2nd col is $\vec{0}$)

Dependence relation $3\vec{a}_2 = \vec{0}$

5. (12 points)

(a) Define what it means for a transformation T to be linear.

T is linear if ① $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T

② $T(cu) = cT(u)$ for all scalars c, u in the domain of T.

(b) Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 2y, -4x)$ is linear using your definition from part (a).

① Let $(x,y), (x',y') \in \mathbb{R}^2$

$T((x,y)+(x',y')) = T(x+x', y+y') = ((x+x') + 2(y+y'), -4(x+x'))$

$= (x+x' + 2y + 2y', -4x - 4x')$

$= (x+2y, -4x) + (x'+2y', -4x') = T(x,y) + T(x',y')$ ✓

② Let $(x,y) \in \mathbb{R}^2$, c a scalar

$T(c(x,y)) = T(cx, cy) = (cx + 2cy, -4(cx)) = (c(x+2y), c(-4x))$

$= c(x+2y, -4y)$

$= cT(x,y)$ ✓ ☒

(c) Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (xy, 0)$ is *not* linear.

Let c be a scalar, $(x,y) \in \mathbb{R}^2$

$T(c(x,y)) = T(cx, cy) = (cx\,cy, 0) = (c^2xy, 0)$

but $cT(x,y) = c(xy, 0) = (cxy, 0)$

In general $c^2xy \neq cxy$ for example let $c = 2$, $x = y = 1$

$4 \cdot 1 \cdot 1 \neq 2 \cdot 1 \cdot 1$

$\therefore$ T is not linear

6. (4 points) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T(x, y) = (2x + y,\ -3x + 5y,\ -x,\ 11y)$.

$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\-3\\-1\\0\end{pmatrix}$ $\quad T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\5\\0\\11\end{pmatrix}$ $\quad$ Matrix $A = \begin{bmatrix}2 & 1\\-3 & 5\\-1 & 0\\0 & 11\end{bmatrix}$

7. (8 points)

(a) Show by means of an example that it is possible to have a set of vectors which spans $\mathbb{R}^3$ but is not linearly independent.

$S = \left\{ \underset{u}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{v}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{w}{\begin{bmatrix}0\\0\\1\end{bmatrix}}, \underset{b}{\begin{bmatrix}2\\0\\0\end{bmatrix}} \right\}$

S spans: $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$ for any $\vec{x}$ in $\mathbb{R}^3$

but S is not LI $2u + 0\cdot v + 0\cdot w - \vec{b} = \vec{0}$ is a dependence relation.

(b) Show by means of an example that is it possible to have set of vectors which is linearly independent but does not span $\mathbb{R}^3$.

$S = \left\{ \underset{\vec{u}}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \right\}$ $\quad \vec{u} \neq \vec{0} \Rightarrow \{\vec{u}\}$ is LI

but clearly S does not span $\mathbb{R}^3$

$\begin{bmatrix}0\\1\\0\end{bmatrix} \notin \operatorname{Span} S$

8. (16 points) Given

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}; \quad \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

$2\times3 \qquad 2\times2 \qquad 2\times1 \qquad 3\times1$

Compute each of the following where possible – if NOT possible, explain why not.

(a) AB

Not defined – # of cols of A ≠ # of rows of B

(b) BA

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -10 & 20 \end{bmatrix}$$

$2\times2 \qquad 2\times3$

(c) $\mathbf{y}\mathbf{x}^{\mathbf{T}}$

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & 0 \\ -8 & 4 \end{bmatrix}$$

$3\times1 \qquad 1\times2$

(d) $\mathbf{y}\mathbf{x}$ ND

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}; \quad \mathrm{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}; \quad \mathrm{y} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Compute each of the following where possible – if NOT possible, explain why not.

(e) A^T

$$\begin{bmatrix} 1 & 0 \\ -1 & -2 \\ 2 & 4 \end{bmatrix}$$

(f) $A^T\mathbf{x}$

$$\begin{bmatrix} 1 & 0 \\ -1 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

3x2 2x1

(g) $\mathbf{y}^{\mathbf{T}}A^T\mathbf{x}$

$$\begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$$

1x3 3x1

9. (8 points) Let $A = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 5 & 6 \\ 10 & -4 & 5 \end{bmatrix}$. Find the inverse of A, if it exists.

(You are not allowed to use any results about determinants.)

$$\left[\begin{array}{ccc|ccc} 2 & -2 & -1 & 1 & 0 & 0 \\ -2 & 5 & 6 & 0 & 1 & 0 \\ 10 & -4 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ +R_1 \\ -5R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & -6 & 10 & -5 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -2R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -9 & -2 & 1 \end{array}\right]$$

NOT invertible

10. (8 points) Let A and B be 3×3 matrices. Suppose A is invertible, but the third column of B is a linear combination of the first two columns of B. Prove, using the definition of matrix-matrix products, that AB is not invertible.

$$AB = A\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix}$$

write $A\vec{b}_3 = A\vec{b}_1 + A\vec{b}_2$

or $\underbrace{A\vec{b}_1 + A\vec{b}_2 - A\vec{b}_3}$ $= \vec{0}$

This is a dependence relation among the columns of AB. Hence the columns of AB are linearly dependent $\Rightarrow$ AB is not invertible