KEY

Tufts University
Department of Mathematics

Math 70
Linear Algebra

February 22, 2016
Exam 1

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must** **show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Please put your name and section number on the last page of this test.
Note there are ten problems and there are problems on every page but this page and the last page.

Problem	Point Value	Points
1	6	
2	20	
3	6	
4	8	
5	20	
6	8	
7	8	
8	8	
9	8	
10	8	
	100	

1. (6 points): Indicate by shading the appropriate box whether each statement is true or false. *For this problem you do not need to give reasons.*

(a) Let A be a 4×3 matrix. Assume there is a unique solution to $Ax = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. Then, there is a unique solution to $A\mathbf{x} = \begin{bmatrix}2\\4\\6\\8\end{bmatrix}$.

T F

$$Ab = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \Leftrightarrow 2(Ab) = 2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}2\\4\\6\\8\end{bmatrix}$$

$$\Leftrightarrow A(2b) = \begin{bmatrix}2\\4\\6\\8\end{bmatrix}$$

b is unique $\Leftrightarrow$ $2b$ is unique

(b) Any set of five vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

T F

Lots of counterexamples:

$S = \{0, u, v, w, x\}$

$S = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\\0\end{pmatrix}, \begin{pmatrix}4\\0\\0\\0\end{pmatrix}, \begin{pmatrix}5\\0\\0\\0\end{pmatrix}\right\}$

(c) Assume A is a 5×4 matrix and assume $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then, the columns of A span $\mathbb{R}^5$.

T F

$Ax = 0 \Rightarrow x = 0$ means A has a pivot position in every column — i.e. 4 pivots. It cannot have a pivot in every row $\Rightarrow$ columns of A do not span $\mathbb{R}^5$.

2. (20 points): Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

(a) Find all values of h such that the system $A\mathbf{x} = \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}$ is consistent.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 3 & 5 & h \\ 0 & 1 & 2 & 1 \end{array}\right] -R_1 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 4 & h-2 \\ 0 & 1 & 2 & 1 \end{array}\right] -\tfrac{1}{2}R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 4 & h-2 \\ 0 & 0 & 0 & 2-\frac{h}{2} \end{array}\right]$$

Need $2 - \frac{h}{2} = 0$ to make sure $Ax = \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}$ is consistent.

$\Rightarrow h = 4$

(b) For each h for which the system in (a) is consistent, how many solutions are there? Why?

There are an infinite # of solutions because $Ax = \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}$ has 1 free variable - namely, x_3 - and we know the system is consistent.

3. (6 points): Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 4x_2 - x_3 \\ x_2 - x_3 \end{bmatrix}.$$ Find the standard matrix A of T.

$$= x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -1 \\ 0 & 1 & -1 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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or: $A = \left(T(e_1) \; T(e_2) \; T(e_3) \right) = \left(T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

4. (8 points): Let A be a 6×6 matrix with five pivot columns.

(a) How many solutions are there to $A\mathbf{x} = \mathbf{0}$? Why?

An infinite # - $Ax=0$ has exactly 1 free variable and since $Ax=0$ is consistent ($x=0$ is a solution) there are an infinite # of solutions.

(b) Is there a solution to $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^6$? Why or why not?

No A has 5 pivot columns so it has pivots in only 5 of 6 rows. The columns of A do <u>not</u> span $\mathbb{R}^6$ so not every equation $Ax=b$ will have a solution.

5. (20 points): Let T be the linear transformation with standard matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 5 \end{bmatrix}$.

(a) (2 points): What is the domain of T? $\mathbb{R}^3$

(b) (2 points): What is the codomain (target) of T? $\mathbb{R}^4$

(c) Is $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ in range(T)? Why or why not?

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 2 & 2 & 5 & 1 \end{array}\right] \begin{array}{c} \\ \\ -R1 \\ -2R1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} \\ \\ -R2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

No - this indicates that the system corresponding to $Ax=b$ is inconsistent - no solution, so $b \notin$ range of T.

(d) Is T one-to-one? Why or why not?

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A has a pivot in every column so $Ax=0 \Rightarrow x=0$ so yes, T is 1-1.

6. (8 points): Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. Prove that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is* linearly dependent.
HINT: You may use facts about pivots or linear equations.

Let $A = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{pmatrix}$ Since A is 2×3, it can have at most 2 pivots so $Ax = 0$ has at least 1 free variable which means $Ax = 0$ has non-0 (non-trivial) solutions. Each one of these nontrivial solutions produces a dependence relation among the columns of A. Thus $\{u, v, w\}$ is linearly dependent.

7. (8 points): Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation. Give the definition for T to be a *linear transformation.*

For all u, v in $\mathbb{R}^n$ and all scalars c:

1) $T(u+v) = T(u) + T(v)$

2) $T(cu) = cT(u)$

8. (8 points): Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Complete the following definitions

(a) T is *one-to-one (injective)* if . . . for any $b \in \mathbb{R}^m$

the equation $T(x) = b$ has at most 1 solution.

(b) T is *onto (surjective)* if . . . for any $b \in \mathbb{R}^m$

the equation $T(x) = b$ has at least one solution.

9. (8 points): Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\mathbb{R}^4$ and assume $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Prove that the set of vectors $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2\}$ is also linearly independent.
HINT: Use the definition of linear independence and that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

pf Suppose $c_1(v_1+v_2) + c_2(v_2) = \theta$. (We must show $c_1 = c_2 = 0$)

Then $c_1v_1 + c_1v_2 + c_2v_2 = \theta$

$c_1v_1 + (c_1+c_2)v_2 = \theta$

Since $\{v_1, v_2\}$ is linearly independent, we know

$c_1 = 0$

$c_1 + c_2 = 0$ But $c_1 = 0 \Rightarrow c_2 = 0$.

$\therefore$ $c_1 = c_2 = 0$ and $\{v_1+v_2, v_2\}$ is linearly independent.

10. (8 points): Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ 3x_2 \end{bmatrix}$. Prove T satisfies the conditions from the definition to be a linear transformation.

HINT: start the proof by letting $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *be vectors in* $\mathbb{R}^2$ *and* $c \in \mathbb{R}$.

pf Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$ and c a scalar

1. $T(u+v) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix}\right)$

$$= \begin{pmatrix} 2(u_1+v_1) + (u_2+v_2) \\ 3(u_2+v_2) \end{pmatrix} = \begin{pmatrix} 2u_1+2v_1 + u_2+v_2 \\ 3u_2+3v_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2u_1+u_2 \\ 3u_2 \end{pmatrix} + \begin{pmatrix} 2v_1+v_2 \\ 3v_2 \end{pmatrix}$$

$$= T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) + T\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T(u)+T(v) \quad \boxtimes$$

2. $T(cu) = T\left(c\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}\right)$

$$= \begin{pmatrix} 2(cu_1) + cu_2 \\ 3cu_2 \end{pmatrix} = \begin{pmatrix} c(2u_1+u_2) \\ c(3u_2) \end{pmatrix}$$

$$= c\begin{pmatrix} 2u_1+u_2 \\ 3u_2 \end{pmatrix} = c\,T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = c\,T(u) \quad \boxtimes$$

Math 70 Exam I February, 22, 2016

Name: ____________________________

Instructor: ____________________________

Section number: ______

I pledge that I have neither given nor received assistance on this exam.

Signature ____________________________