KEY

Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics
Exam I

October 7, 2013
Section 1

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices must be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	20	
3	12	
4	6	
5	10	
6	8	
7	9	
8	8	
9	10	
10	7	
	100	

1. (10 pts) **True/false questions.** Decide whether each of the statements below is true or false. Indicate your answer by shading the appropriate box. No partial credit.

   (a) If $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. $\boxed{T}$ $\boxed{F}$

   (b) Every matrix transformation is linear. $\boxed{T}$ $\boxed{F}$

   (c) If A is a 3×2 matrix, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one. $\boxed{T}$ $\boxed{F}$

   $A = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$ $\quad T : \mathbb{R}^2 \Rightarrow \mathbb{R}^3$

   $T(\vec{x}) = A\vec{x}$

   (d) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector x in $\mathbb{R}^n$ maps to a unique vector y in $\mathbb{R}^m$. $\boxed{T}$ $\boxed{F}$

   (this is the definition of a function)

   (e) If $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \vec{0}$. $\boxed{T}$ $\boxed{F}$

2. (20 pts) Consider the system of equations:

$$5x_1 - 5x_2 + 10x_3 + 15x_5 = 20$$

$$2x_1 - 2x_2 + 4x_3 + x_4 + 2x_5 = 3$$

(a) Write down the corresponding vector equation.

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$$x_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 10 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 15 \\ 2 \end{bmatrix} = \begin{bmatrix} 20 \\ 3 \end{bmatrix}$$

(b) Write down the corresponding matrix equation and label it $A\mathbf{x} = \mathbf{b}$.

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$$\begin{bmatrix} 5 & -5 & 10 & 0 & 15 \\ 2 & -2 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 20 \\ 3 \end{bmatrix}$$

(c) The linear transformation T with $T(x) = Ax$ represents a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ where

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$n =$ 5 and $m =$ 2.

(8)

(d)

$$5x_1 - 5x_2 + 10x_3 + 15x_5 = 20$$
$$2x_1 - 2x_2 + 4x_3 + x_4 + 2x_5 = 3$$

Find all solutions to the system and write your answer in parametric vector form.

$$\left[\begin{array}{ccccc|c} 5 & -5 & 10 & 0 & 15 & 20 \\ 2 & -2 & 4 & 1 & 2 & 3 \end{array}\right] \left(\tfrac{1}{5}\right) \sim \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 4 \\ 2 & -2 & 4 & 1 & 2 & 3 \end{array}\right] -2R_1$$

$$-2 \quad 2 \quad -4 \quad 0 \quad -6 \quad -8$$

$$\begin{array}{cccccc} & f & f & & f & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & -4 & -5 \end{array}\right]$$

$x_1 - x_2 + 2x_3 + 3x_5 = 4$

$x_2 = x_2$

$x_3 = x_3$

$x_4 \quad - 4x_5 = -5$

$x_5 = x_5$

$x_1 = 4 + x_2 - 2x_3 - 3x_5$

$x_2 = x_2$

$x_3 = x_3$

$x_4 = -5 + 4x_5$

$x_5 = x_5$

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ -5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

or

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ -5 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

3. (12 pts) Fill in the blanks.
   (a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *onto* if for every **b** in $\mathbb{R}^m$ there is at least one **x** in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

   If T is onto and $T(\mathbf{x}) = A\mathbf{x}$ the columns of A Span $\mathbb{R}^m$

   and A has a pivot position in every row.

   (b) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *one-to-one* if for every **b** in $\mathbb{R}^m$

   there is at most one **x** in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

   If T is one-to-one and $T(x) = Ax$ the columns of A are linearly independent

   and A has a pivot position in every column.

4. (6 pts)
   (a) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^m$ is an onto linear transformation. List all possible values for m.

   $m = 1, 2, 3, 4.$

   (b) Suppose $T : \mathbb{R}^n \to \mathbb{R}^6$ is a one-to-one linear transformation. List all possible values for n.

   $n = 1, 2, 3, 4, 5, 6$

5. (10 points) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. Show that if c is a scalar then $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

$$c(\vec{u}+\vec{v}) = c\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right)$$

$$= c\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

$$= \begin{bmatrix} c(u_1+v_1) \\ c(u_2+v_2) \\ \vdots \\ c(u_n+v_n) \end{bmatrix}$$

$$= \begin{bmatrix} cu_1+cv_1 \\ cu_2+cv_2 \\ \vdots \\ cu_n+cv_n \end{bmatrix}$$

$$= \begin{bmatrix} c_1u_1 \\ c_2u_2 \\ \vdots \\ c_nv_n \end{bmatrix} + \begin{bmatrix} c_1v_1 \\ c_2v_2 \\ \vdots \\ c_nv_n \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + c\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = c\vec{u} + c\vec{v} \quad \boxtimes$$

6. (8 pts) Show why **neither** of the following transformations is linear. Be as explicit as possible.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x - 2, z)$.

Three possible answers ① $T(0,0,0) = (0-2, 0) = (-2,0) \neq 0_{\mathbb{R}^2}$

Since T linear $\Rightarrow T(0) = 0$, T cannot be linear.

② $T((x,y,z) + (x',y',z')) = T(x+x', y+y', z+z') = ((x+x') - 2, z+z')$

$T(x,y,z) + T(x',y',z') = (x-2, z) + (x'-2, z') = ((x+x') - 4, z+z')$

Since $(x+x') - 2 \neq (x+x') - 4$ in general, for example $x = 1, x' = 1$,

T does not preserve addition $\Rightarrow T$ is not linear

③ Let $\vec{x} = (x,y,z) = (1,1,1)$, $c = 2$. Then $T(c\vec{x}) = T(2,2,2) = (2-2, 2) = (0,2)$

$cT(\vec{x}) = 2T(1,1,1) = 2(-1,1) = (-2,2)$

$T(c\vec{x}) \neq cT(\vec{x}) \Rightarrow T$ is not linear

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x, xy)$.

There are also different answers possible here.

① T does not preserve addition:

$T((1,1) + (2,2)) = T(3,3) = (3,9)$

$T(1,1) + T(2,2) = (1,1) + (2,4) = (3,5)$

$\Big\}$ $T((1,1)+(2,2)) \neq T(1,1) + T(2,2)$

or

$T((x,y) + (x',y')) = T(x+x', y+y') = (x+x', (x+x')(y+y'))$

$T(x,y) + T(x',y') = (x, xy) + (x', x'y') = (x+x', xy + x'y')$

$(x+x')(y+y') \neq xy + x'y'$ in general, <u>for example</u> let $x = y = x' = y' = 1$ then $4 \neq 2$

② T does not preserve scalar multiplication:

$T(c(x,y)) = T(cx, cy) = (cx, c^2xy)$

$cT(x,y) = c(x, xy) = (cx, cxy)$

$\Big\}$ Since $c^2 \neq c$ in general, for example $c = 2$, T does not preserve scalar multiplication

So T is not linear

7. (9 pts) Determine whether each of the following sets in $\mathbb{R}^3$ is linearly independent. No explanation needed. Indicate your answer by shading the appropriate box. No partial credit.

(a) $S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Yes No

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$\vec{u}$ $\quad \vec{u} \neq \theta$

(b) $S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} \right\}$. Yes No

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$\{\vec{u}, \vec{v}\}$ $\quad \vec{v} = -3\vec{u}$

(c) $S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. Yes No

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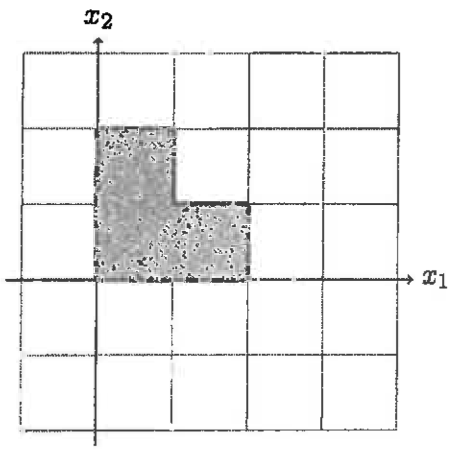
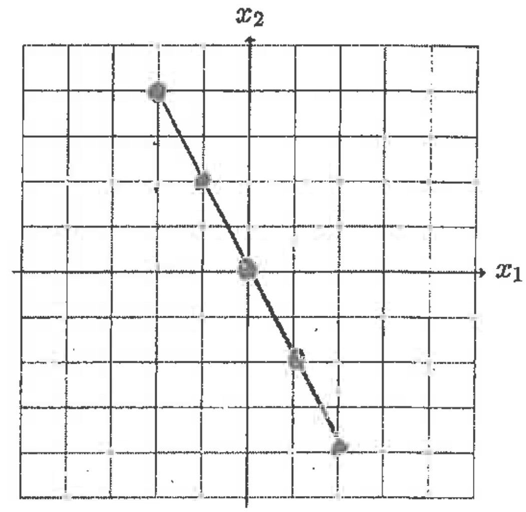
8. (8 points)

(a) Here is the matrix of a linear transformation: $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

Draw the image of the figure below under the transformation $T(\mathbf{x}) = A\mathbf{x}$..

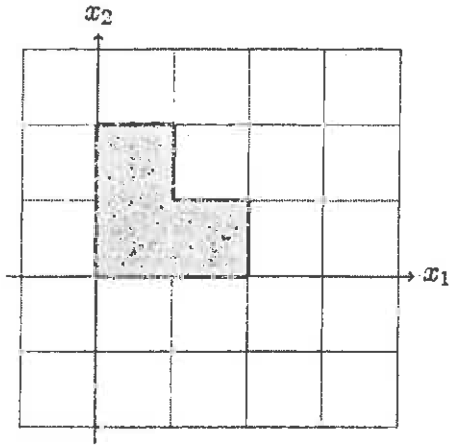


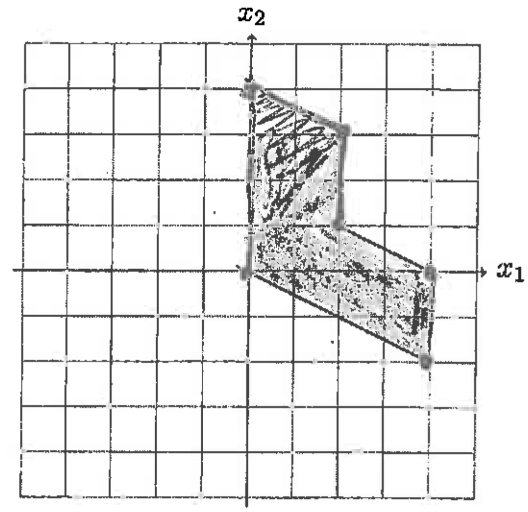


$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 & 0 & 1 & 2 \\ 0 & 4 & 2 & 0 & -2 & -4 \end{bmatrix}$$

(b) Here is the matrix of a linear transformation: $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$.

Draw the image of the figure below under the transformation.





$$\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 2 & 2 & 0 \\ 0 & -2 & 0 & 1 & 3 & 4 \end{bmatrix}$$

9. (10 points) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation.

(a) Define the standard matrix A of T.

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$A = [T(\vec{e}_1) \ T(\vec{e}_2)]$ or $A = \left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]$

(b) Suppose that

$$T\left(\begin{bmatrix}3\\0\end{bmatrix}\right) = \begin{bmatrix}-12\\6\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\5\end{bmatrix}$$

Use your definition from part (a) along with the definition of a linear transformation to find the standard matrix A of T.

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$\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}3\\0\end{bmatrix}$

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\frac{1}{3}\begin{bmatrix}3\\0\end{bmatrix}\right) = \frac{1}{3}T\left(\begin{bmatrix}3\\0\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}-12\\6\end{bmatrix} = \begin{bmatrix}-4\\2\end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} - \begin{bmatrix}3\\-1\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}3\\0\end{bmatrix} - \begin{bmatrix}3\\-1\end{bmatrix}\right)$

$= T\left(\begin{bmatrix}3\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)$

$= \begin{bmatrix}-12\\6\end{bmatrix} - \begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}-14\\1\end{bmatrix}$

$A = \begin{bmatrix}-4 & -14\\2 & 1\end{bmatrix}$

10. (7 points) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ span $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \vec{0}$, for $i = 1, 2, 3, 4$. Show that T is the zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^3$, then $T(\mathbf{x}) = \vec{0}$. (Your answer should not involve matrices.)

Let $\vec{x}$ be any vector in $\mathbb{R}^3$. Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ spans $\mathbb{R}^3$,

$\exists$ scalars c_1, c_2, c_3, c_4 with

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

Apply T to each side of the equation:

$$T(\vec{x}) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) + c_4T(\vec{v}_4)$$

$$T(\vec{x}) = c_1 0 + c_2 0 + c_3 0 + c_4 0$$

$$T(\vec{x}) = 0$$

Since $\vec{x}$ was chosen arbitrarily, T is the zero map:

$T(\vec{x}) = 0$ for all $x \in \mathbb{R}^3$. $\boxtimes$