Math 70 Exam I Spring 2016

Let $\{\mathbf{v_1}, \mathbf{v_2}\}$ be two vectors in $\mathbb{R}^4$. Prove that if $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent then $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ is also is linearly independent. (Note: The same exact proofs below work with $\mathbb{R}^4$ replaced by any vector space $\mathbf{V}$.)

1. Direct Proof (This is by far the simplest and most clear proof !)

   Suppose $x_1(\mathbf{v_1} + \mathbf{v_2}) + x_2\mathbf{v_2} = \vec{0}$. (We must show that $x_1 = x_2 = 0$.)
   Then $x_1\mathbf{v_1} + x_1\mathbf{v_2} + x_2\mathbf{v_2} = \vec{0}$.
   So $x_1\mathbf{v_1} + (x_1 + x_2)\mathbf{v_2} = \vec{0}$.
   Since $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent we know $x_1 = 0$ and $x_1 + x_2 = \vec{0}$. But $x_1 = 0 \to x_2 = 0$. ■

2. Indirect Proof (Using the contrapositive)
   (Show that if $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ is linearly dependent, then $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent.)

   Suppose $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ is linearly dependent.
   Then there exist scalars c_1 and c_2, not both 0, with $c_1(\mathbf{v_1} + \mathbf{v_2}) + c_2\mathbf{v_2} = \vec{0}$.
   Then $c_1\mathbf{v_1} + c_1\mathbf{v_2} + c_2\mathbf{v_2} = \vec{0}$.
   So $c_1\mathbf{v_1} + (c_1 + c_2)\mathbf{v_2} = \vec{0}$.

   We know that c_1 and c_2 are not both 0.
   If $c_1 \neq 0$ then $c_1\mathbf{v_1} + (c_1 + c_2)\mathbf{v_2} = \vec{0}$ is a dependence relation so $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent.

   If $c_1 = 0$ then we know $c_2 \neq 0 \to c_1 + c_2 \neq 0$. Once again $c_1\mathbf{v_1} + (c_1 + c_2)\mathbf{v_2} = \vec{0}$ is a dependence relation which means $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent. ■

3. Proof by Contradiction.

   Suppose $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent but $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ is linearly dependent.
   Since $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ is linearly dependent there exist scalars c_1 and c_2, not both 0, with
   $c_1(\mathbf{v_1} + \mathbf{v_2}) + c_2\mathbf{v_2} = \vec{0}$.
   Then $c_1\mathbf{v_1} + c_1\mathbf{v_2} + c_2\mathbf{v_2} = \vec{0}$.
   So $c_1\mathbf{v_1} + (c_1 + c_2)\mathbf{v_2} = \vec{0}$.

   We know that c_1 and c_2 are not both 0.
   If $c_1 \neq 0$ then $c_1\mathbf{v_1} + (c_1 + c_2)\mathbf{v_2} = \vec{0}$ is a dependence relation so this would make $\{\mathbf{v_1}, \mathbf{v_2}\}$ linearly dependent. This is a contradiction because we had assumed that $\{\mathbf{v_1}, \mathbf{v_2}\}$ was linearly independent. Hence $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ cannot be linearly dependent and so must be linearly independent.

   If $c_1 = 0$ then we know $c_2 \neq 0 \to c_1 + c_2 \neq 0$. Once again $c_1\mathbf{v_1} + (c_1 + c_2)\mathbf{v_2} = \vec{0}$ is a dependence relation which means $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent. This is a contradiction because we had assumed that $\{\mathbf{v_1}, \mathbf{v_2}\}$ was linearly independent. Hence $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ cannot be linearly dependent and so must be linearly independent. ■

4. Some students tried to do the proof as follows: If $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent, then $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ is linearly independent because $\mathbf{v_1} + \mathbf{v_2}$ and $\mathbf{v_2}$ cannot be multiples of each other. Most of the students who did this did not consider both cases. This method is available only because we are dealing with exactly 2 vectors. It does not generalize to three or more vectors. Furthermore, it is ugly!

   Assume that $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent. We will show that $\mathbf{v_1} + \mathbf{v_2}$ and $\mathbf{v_2}$ cannot be multiples of each other and hence $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ cannot be linearly dependent and so must be linearly independent.

   (a) Case 1. Suppose there is a scalar c with $\mathbf{v_1} + \mathbf{v_2} = c\mathbf{v_2}$.
   Then $\mathbf{v_1} = c\mathbf{v_2} - \mathbf{v_2}$, so $\mathbf{v_1} = (c-1)\mathbf{v_2}$. This cannot happen because $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.

   (b) Case 2. Suppose there is a scalar d with $\mathbf{v_2} = d(\mathbf{v_1} + \mathbf{v_2})$.
      (i) Subcase 1. If $d = 0$ then $\mathbf{v_2} = \vec{0}$ and this cannot happen since $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.
      (ii) Subcase 2. If $d \neq 0$ then $\mathbf{v_2} = d\mathbf{v_1} + d\mathbf{v_2}$ or $\mathbf{v_2} - d\mathbf{v_2} = d\mathbf{v_1}$ which becomes $(1-d)\mathbf{v_2} = d\mathbf{v_1}$.
      Solving for $\mathbf{v_1}$ we get: $(\frac{1-d}{d})\mathbf{v_2} = \mathbf{v_1}$. This cannot happen because $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.

   Since we have shown that $\mathbf{v_1} + \mathbf{v_2}$ and $\mathbf{v_2}$ cannot be multiples of each other, we have proven that $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2}\}$ is linearly independent. ■

5. Common Wrong Proof

   Since $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent we know that the only solution to $x_1\mathbf{v_1} + x_2\mathbf{v_2} = \vec{0}$ is $x_1 = x_2 = 0$. Now consider

$$(*) \qquad x_1(\mathbf{v_1} + \mathbf{v_2}) + x_2\mathbf{v_2} = \vec{0}$$

   We know that $x_1 = x_2 = 0$ and so we are done. THIS IS TOTALLY BACKWARDS. You cannot begin by knowing $x_1 = x_2 = 0$. When you set up equation (*) you have to assume that you know nothing about the scalars. Instead of using x_1 and x_2, rewrite equation (*) as follows:

$$(**) \qquad y_1(\mathbf{v_1} + \mathbf{v_2}) + y_2\mathbf{v_2} = \vec{0}$$

   Since y_1 and y_2 could be any scalars you CANNOT assume they are both 0. You must reach that conclusion only after rewriting equation ** and then using what you know about $\mathbf{v_1}$ and $\mathbf{v_2}$. (See the direct proof.)

6. Common mistakes
   (a) Some students distributed incorrectly at step #1 of the direct proof and got: $x_1\mathbf{v_1} + 2x_2\mathbf{v_2} = \vec{0}$.
   This was incorrect and simplified the proof. (-3)

   (b) Some students got to the equation $x_1\mathbf{v_1} + x_1\mathbf{v_2} + x_2\mathbf{v_2} = \vec{0}$ in the direct proof and then, ignoring the middle term on the left, said that since $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent, $x_1 = x_2 = 0$. This was a major error. (-5)