**Section 1.9 The Matrix of a Linear Transformation**

Key Concepts

- Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.
- The action of $T$ on unit vectors $\mathbf{e}_i, i = 1 \ldots m$, determines the structure of $A$.
- Important properties of $T$ (one-to-one, onto) are intimately related to known properties of $A$.

**Example** Given unit vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$

any $\mathbf{x} \in \mathbb{R}^m$ can be written as

$$\begin{aligned} \mathbf{x} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m \\ &= \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{I_m} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{\mathbf{x}} \\ &= I_m\mathbf{x}. \end{aligned}$$

Recall from section 1.8: if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

General result:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p).$$

**Example** Suppose $T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ with

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 5 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } T(\mathbf{e}_3) = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 7 \end{bmatrix}.$$

Compute $T(\mathbf{x})$ for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ and determine the matrix $A$ that implements the transformation $T$.

$$A = [T(e_1) \;\; T(e_2) \;\; T(e_3)] = \begin{bmatrix} 2 & 5 & -1 \\ -3 & 0 & -2 \\ 4 & 1 & 0 \\ 5 & -1 & 7 \end{bmatrix}$$

So, $T(x) = Ax = \begin{bmatrix} 2 & 5 & -1 \\ -3 & 0 & -2 \\ 4 & 1 & 0 \\ 5 & -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= \begin{bmatrix} 2x_1 + 5x_2 - x_3 \\ -3x_1 + 0x_2 - 2x_3 \\ 4x_1 + x_2 + 0x_3 \\ 5x_1 - x_2 + 7x_3 \end{bmatrix}$$

**Question:** What is the size of the matrix $A$ that corresponds to the map $T : \mathbb{R}^p \to \mathbb{R}^q$?

$T: \underset{\mathbb{R}^p}{\underset{\cap}{x}} \longmapsto \underset{\mathbb{R}^q}{\underset{\cap}{Ax}}$ so $A$ is a $q \times p$ matrix

**Theorem 10** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix $A$ such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

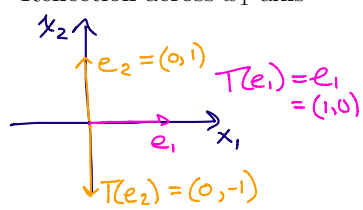
In fact, $A$ is the $m \times n$ matrix whose $j$th column is the vector $T(\mathbf{e}_j)$, with $\mathbf{e}_j \in \mathbb{R}^n$:

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots T(\mathbf{e}_n)]$$

The matrix $A$ is called the **standard matrix for the linear transformation** $T$.

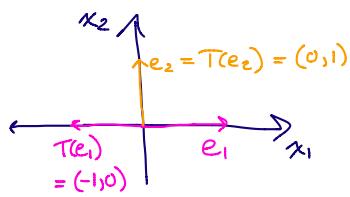
**Example** Determine the standard matrices for the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$.

Reflection across $x_1$ axis



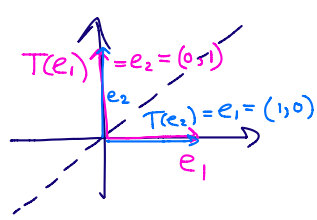

So $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the standard matrix for $T$.

Reflection across $x_2$ axis



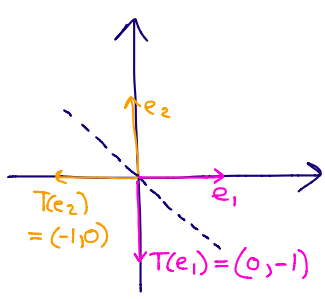

So $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is the standard matrix for $T$.

Reflection across line $x_2 = x_1$




So $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix for $T$.

Reflection across line $x_2 = -x_1$



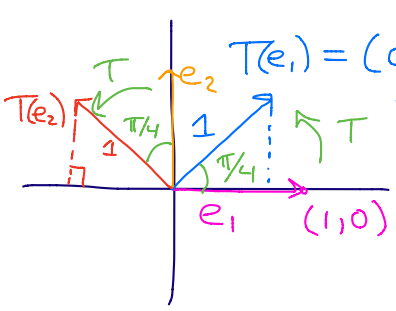

So $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is the standard matrix for $T$.

**Example** Find the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ if $T : \mathbf{x} \mapsto \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$.

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 0x_3 \\ 4x_1 + 0x_2 + 0x_3 \\ 3x_1 + 2x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ is the standard matrix for $T$.

**Example** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates each point in $\mathbb{R}^2$ about the origin through and angle $\pi/4$ radians (counterclockwise). Determine the standard matrix for $T$.



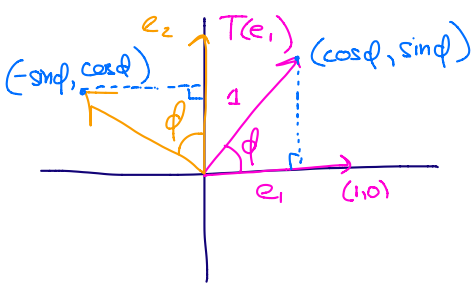

$T(e_1) = (\cos(\pi/4), \sin \pi/4)$
$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$T(e_2) = (\cos(3\pi/4), \sin(3\pi/4))$
$= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

So, $A = [T(e_1) \;\; T(e_2)]$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$ is the standard matrix for $T$.

**Question:** Determine the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point in $\mathbb{R}^2$ counterclockwise around the origin through an angle of $\phi$ radians.




So $A = [T(e_1) \;\; T(e_2)]$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$ is the standard matrix for $T$.

Definitions:

- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

  Equivalent ways to state this definition are:

  For each $b \in \mathbb{R}^m$, there is at least one $x \in \mathbb{R}^n$ such that $T(x) = b$.

  Note: $T$ is *not* onto $\mathbb{R}^m$ when there is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution, or equivalently, whenever the is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the system $[A|\mathbf{b}]$ is inconsistent.

- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

For each $b \in \mathbb{R}^m$, there is at most one $x \in \mathbb{R}^n$ such that $T(x) = b$.

**Example** Let $T$ be the transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 3 & -4 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Does $T$ map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is $T$ a one-to-one mapping? Explain your reasoning.

$T$ is onto $\mathbb{R}^3 \iff$ For each $b \in \mathbb{R}^3$, there is at least one $x \in \mathbb{R}^4$ such that $T(x) = b$.

$\iff$ For each $b \in \mathbb{R}^3$, $T(x) = b$ has at least one solution.

$\iff$ Every $b \in \mathbb{R}^3$ is in the span of the columns of $A$.

$\iff$ The columns of $A$ span $\mathbb{R}^3$ (or, use Thm 12(a) below to get to this point)

$A$ spans $\mathbb{R}^3 \iff A$ has a pivot position in each row.

$A$ does have a pivot position in each row (seen from echelon form), so yes, $A$ is onto.

$T$ is one-to-one $\iff$ For every $b \in \mathbb{R}^3$, $Ax=b$ has at most one soln.

$A$ has a free variable, $x_3$, so $Ax=0$ has infinitely many solutions (which is more than one).

So, $A$ is not one-to-one.

**Theorem 11** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Proof:

"$\Rightarrow$" Suppose $T$ is one-to-one. Since $T$ is linear, $T(0) = 0$. Since $T$ is one-to-one, $Tx = 0$ can't have any other solutions. So the only soln to $Tx = 0$ is the trivial soln. $\blacksquare$

"$\Leftarrow$" Consider the contrapositive. Suppose $T$ is <u>not</u> one to one. Then there are some $u, v \in \mathbb{R}^n$, $u \neq v$, such that $T(u) = T(v)$. Then $T(u-v) = T(u) - T(v)$ because $T$ is linear
$= 0$ since $T(u) = T(v)$

So, $T(u-v) = 0$, but $u - v \neq 0$ since $u \neq v$. So $Tx = 0$ has a nontrivial soln. $\blacksquare$

**Theorem 12** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:

a. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$.

b. $T$ is one-to-one if and only if the columns of $A$ are linearly independent.

Proof:

(a) $T$ is onto $\mathbb{R}^m$

$\Leftrightarrow$ For each $b \in \mathbb{R}^m$, there is at least one $x \in \mathbb{R}^n$ such that $T(x) = b$.

$\Leftrightarrow$ For each $b \in \mathbb{R}^m$, $T(x) = b$ has at least one solution.

$\Leftrightarrow$ Every $b \in \mathbb{R}^m$ is in the span of the columns of $A$

$\Leftrightarrow$ The columns of $A$ span $\mathbb{R}^m$

(b) $T$ is one-to-one $\Leftrightarrow$ $Tx = 0$ has only the trivial soln (by above)

$\Leftrightarrow$ The columns of $A$ are linearly indep (§1.7).

Summary of section 1.9

- Each $T : \mathbb{R}^n \to \mathbb{R}^m$ corresponds to standard matrix $A$ of size $m \times n$.
- The standard matrix is determined by the action of $T$ on the standard unit vectors $\mathbf{e}_i$.
- Properties of $T$ (onto, one-to-one) are deeply related to questions of existence/uniqueness of solutions to $A\mathbf{x} = \mathbf{b}$.