1.8 Introduction to Linear Transformations

Another way to view $A\mathbf{x} = \mathbf{b}$:

Matrix A is an object acting on $\mathbf{x}$ by multiplication to produce a new vector $A\mathbf{x}$ or $\mathbf{b}$.

EXAMPLE:

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Suppose A is $m \times n$. Solving $A\mathbf{x} = \mathbf{b}$ amounts to finding all ____ in $\mathbf{R}^n$ which are transformed into vector $\mathbf{b}$ in $\mathbf{R}^m$ through multiplication by A.

transformation
"machine"

Matrix Transformations

A **transformation** T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$.

$$T : \mathbf{R}^n \to \mathbf{R}^m$$

Terminology:

$\mathbf{R}^n$: **domain** of T $\mathbf{R}^m$: **codomain** of T

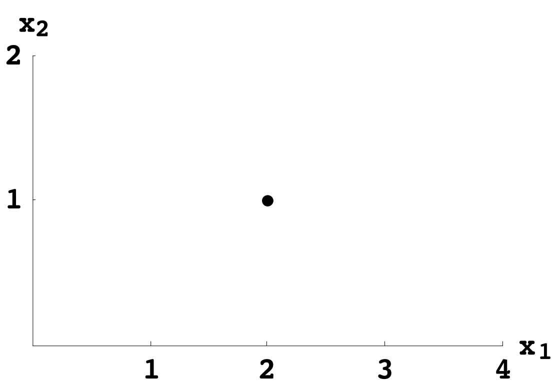
$T(\mathbf{x})$ in $\mathbf{R}^m$ is the **image** of $\mathbf{x}$ under the transformation T

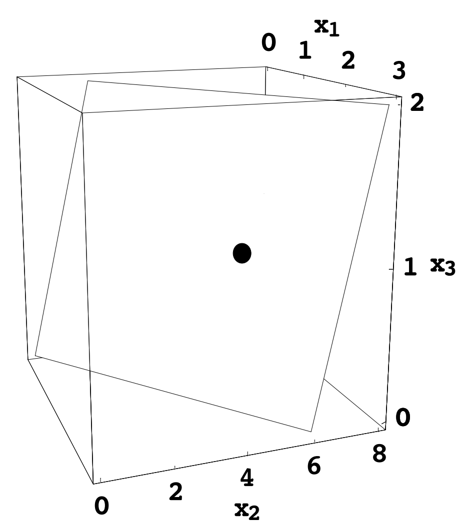
Set of all images $T(\mathbf{x})$ is the **range** of T

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

Then if $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$





EXAMPLE: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then define a transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find an $\mathbf{x}$ in $\mathbf{R}^3$ whose image under T is $\mathbf{b}$.
b. Is there more than one $\mathbf{x}$ under T whose image is $\mathbf{b}$. *(uniqueness problem)*
c. Determine if $\mathbf{c}$ is in the range of the transformation T. *(existence problem)*

Solution: (a) Solve $T(x)$ = b for $\mathbf{x}$.

I.e., solve Ax = b or

$$\begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -5 & 10 & -15 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 2x_2 - 3x_3 + 2$$

x_2 is free

x_3 is free

Let $x_2 =$ 1 and $x_3 =$ 1. Then $x_1 =$ 1.

$$\text{So } \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Is there an **x** for which $T(\mathbf{x}) = \mathbf{b}$?

Free variables exist

$\Downarrow$

There is more than one **x** for which $T(\mathbf{x}) = \mathbf{b}$

(c) Is there an **x** for which $T(\mathbf{x}) = \mathbf{c}$? This is another way of asking if $A\mathbf{x} = \mathbf{c}$ is Consistent.

Augmented matrix:

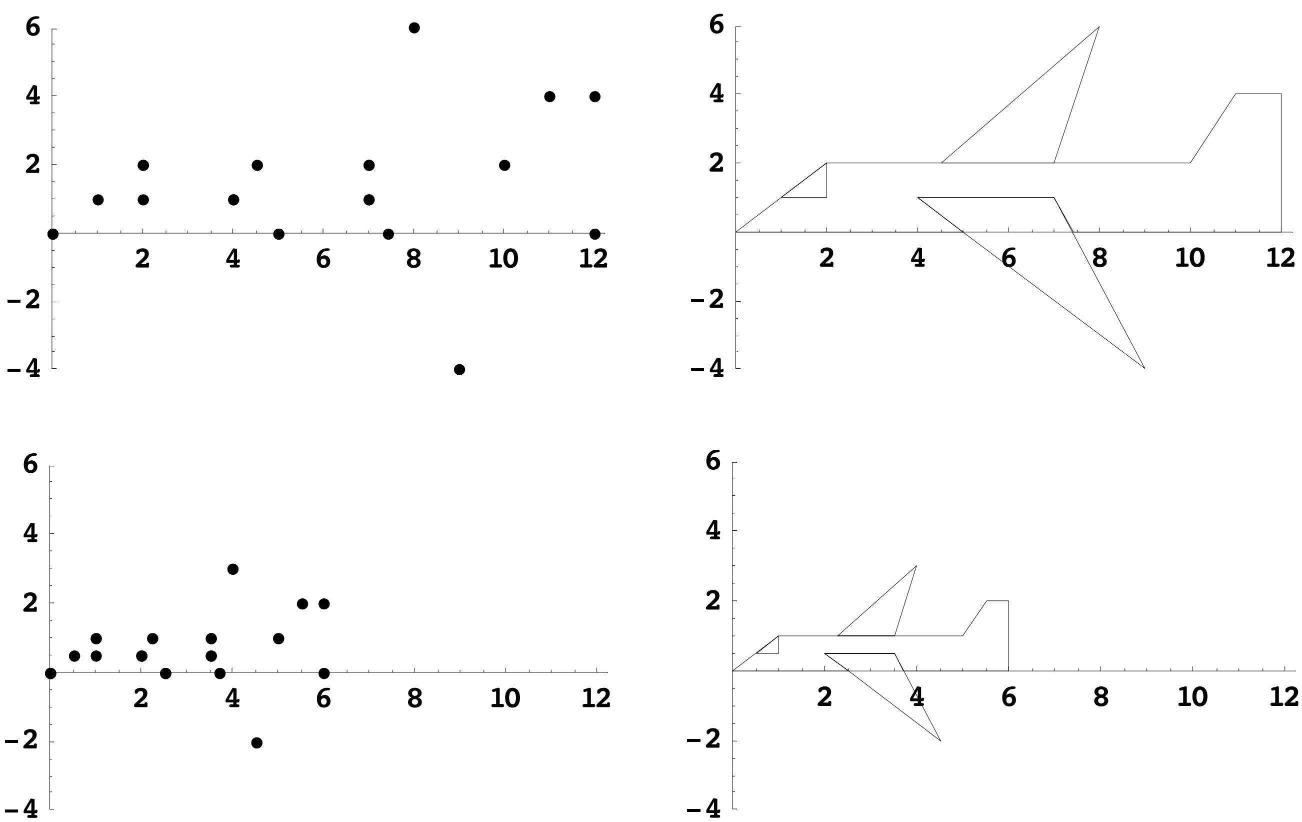
$$\begin{bmatrix} 1 & -2 & 3 & 3 \\ -5 & 10 & -15 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c is not in the image of T.

Matrix transformations have many applications - including *computer graphics*.

EXAMPLE: Let $A = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$. The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is an example of a **contraction** transformation. The transformation $T(\mathbf{x}) = A\mathbf{x}$ can be used to move a point $\mathbf{x}$.

$$\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \qquad T(\mathbf{u}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Linear Transformations

If A is $m \times n$, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the following properties:

$$T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = \underline{\;Au\;} + \underline{\;Av\;}$$

$$= \underline{\;T(u)\;} + \underline{\;T(v)\;}$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = \underline{\;c\;}A\mathbf{u} = \underline{\;c\;}T(\mathbf{u})$$

for all $\mathbf{u}$,$\mathbf{v}$ in $\mathbf{R}^n$ and all scalars c.

DEFINITION

A transformation T is **linear** if:

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T.

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in the domain of T and all scalars c.

Every matrix transformation is a **linear** transformation.

RESULT If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \qquad \text{and} \qquad T(c\mathbf{u} + d\mathbf{v}) = c\mathbf{T}(\mathbf{u}) + d\mathbf{T}(\mathbf{v}).$$

Proof:

$$T(\mathbf{0}) = T(0\mathbf{u}) = \underline{\ 0\ }\,T(\mathbf{u}) = \underline{\ 0\ }.$$

$$T(c\mathbf{u} + d\mathbf{v}) = T(\ cu\ ) + T(\ dv\ )$$

$$= \underline{\ c\ }\,T(\ u\ ) + \underline{\ d\ }\,T(\ v\ )$$

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation which maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: First, note that

$$T(\mathbf{e}_1) = \underline{\quad y_1 \quad} \qquad \text{and} \qquad T(\mathbf{e}_2) = \underline{\quad y_2 \quad}.$$

Also

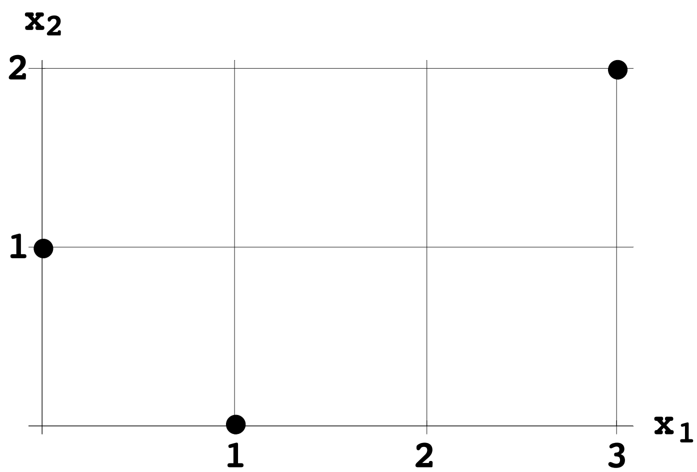
$$\underline{\ 3\ }\,\mathbf{e}_1 + \underline{\ 2\ }\,\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

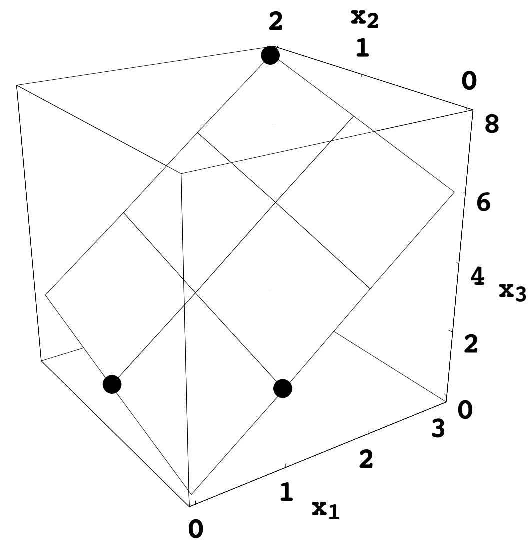
Then

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T(\underline{\ 3\ }\,\mathbf{e}_1 + \underline{\ 2\ }\,\mathbf{e}_2) =$$

$$\underline{\ 3\ }\,T(\mathbf{e}_1) + \underline{\ 2\ }\,T(\mathbf{e}_2) = 3y_1 + 2y_2$$

$$= \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}$$





$$T(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3T(\mathbf{e}_1) + 2T(\mathbf{e}_2)$$

Also

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\underline{\ x_1\ }\mathbf{e}_1 + \underline{\ x_2\ }\mathbf{e}_2) =$$

$$\underline{\ x_1\ }T(\mathbf{e}_1) + \underline{\ x_1\ }T(\mathbf{e}_2) = x_1 y_1 + x_2 y_2$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

EXAMPLE: Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(x_1, x_2, x_3) = (|x_1 + x_3|, 2 + 5x_2)$. Show that T is a not a linear transformation.

Solution: Another way to write the transformation:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$$

Provide a **counterexample** - example where $T(\mathbf{0}) = \mathbf{0}$, $T(c\mathbf{u}) = c\mathbf{T}(\mathbf{u})$ or $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ is violated.

A counterexample:

$$T(\mathbf{0}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \underline{\vec{0}}$$

which means that T is not linear.

Another counterexample: Let $c = -1$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$T(c\mathbf{u}) = T\left(\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} |-1 + -1| \\ 2 + 5(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and

$$cT(\mathbf{u}) = -1T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = -1\begin{bmatrix} |1| \\ 2+5(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Therefore $T(c\mathbf{u}) \neq$ __c__ $T(\mathbf{u})$ and therefore T is

not __linear______________.