1.7 Linear Independence

A homogeneous system such as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The vector equation has the trivial solution ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$), but is this the *only solution*?

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

$\uparrow$

linear dependence relation

(when weights are not all zero)

EXAMPLE Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable $\Rightarrow$ there are nontrivial solutions.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set

(b) Reduced echelon form: $\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -33x_3 \\ x_2 &= 18x_3 \\ x_3 &= x_3 \end{aligned}$

Let $x_3 =$ __2__ (any nonzero number).

Then $x_1 =$ __−66__ and $x_2 =$ __36__.

$$\underline{-66}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \underline{36}\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \underline{2}\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\underline{-66}\,\mathbf{v}_1 + \underline{36}\,\mathbf{v}_2 + \underline{2}\,\mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

Linear Independence of Matrix Columns

A linear dependence relation such as

$$-33\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be written as the matrix equation:

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}\begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Special Cases

Sometimes we can determine linear independence of a set with minimal effort.

1. *A Set of One Vector*

Consider the set containing one nonzero vector: $\{\mathbf{v}_1\}$

The only solution to $x_1\mathbf{v}_1 = 0$ is $x_1 =$ __0__.

$\boxed{\text{So } \{\mathbf{v}_1\} \text{ is linearly independent when } \mathbf{v}_1 \neq \mathbf{0}.}$

2. *A Set of Two Vectors*

EXAMPLE Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

a. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set or a linearly independent set.

b. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that $\mathbf{u}_2 = \underline{\quad 2 \quad}\mathbf{u}_1$. Therefore

$$\underline{\quad 2 \quad}\mathbf{u}_1 + \underline{\quad -1 \quad}\mathbf{u}_2 = 0$$

This means that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly __dependent__ set.

(b) Suppose (for v_1, v_2 given above)

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}.$$

Then $\mathbf{v}_1 = \dfrac{-d}{c}\mathbf{v}_2$ if $c \neq 0$. But this is impossible since $\mathbf{v}_1$ is

__not__ a multiple of $\mathbf{v}_2$ which means $c =$ __0__.

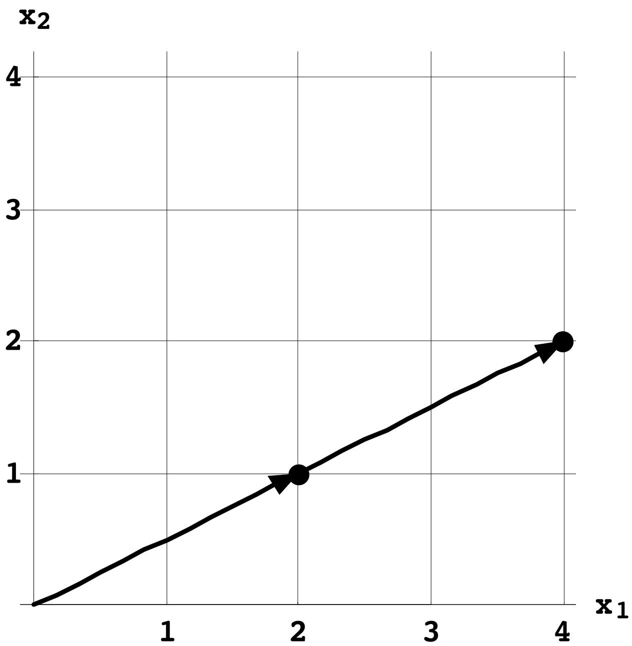
Similarly, $\mathbf{v}_2 = \dfrac{-c}{d}\mathbf{v}_1$ if $d \neq 0$.

But this is impossible since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and so $d = 0$.

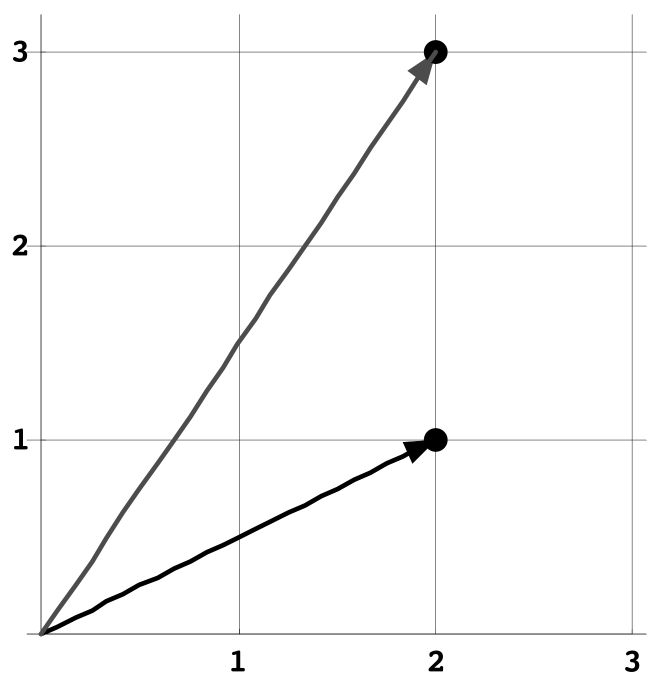
This means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly __independent__ set.

A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.



linearly dependent



linearly independent

3. *A Set Containing the* **0** *Vector*

Theorem 9

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\mathbf{v}_1 = \underline{\ 0\ }$. Then

$$\underline{\ 1\ }\mathbf{v}_1 + \underline{\ 0\ }\mathbf{v}_2 + \cdots + \underline{\ 0\ }\mathbf{v}_p = \mathbf{0}$$

which shows that S is linearly dependent.

4. *A Set Containing Too Many Vectors*

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

Outline of Proof:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \text{ is } n \times p$$

Suppose $p > n$.

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has more variables than equations

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has nontrivial solutions

$\Rightarrow$columns of A are linearly dependent

EXAMPLE With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

indep, b/c neither vector is a scalar multiple of the other.

b. Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$

Dependent, b/c more vectors than there are entries in each vector.

c. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

dependent,
b/c contains the zero vector.

d. $\left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

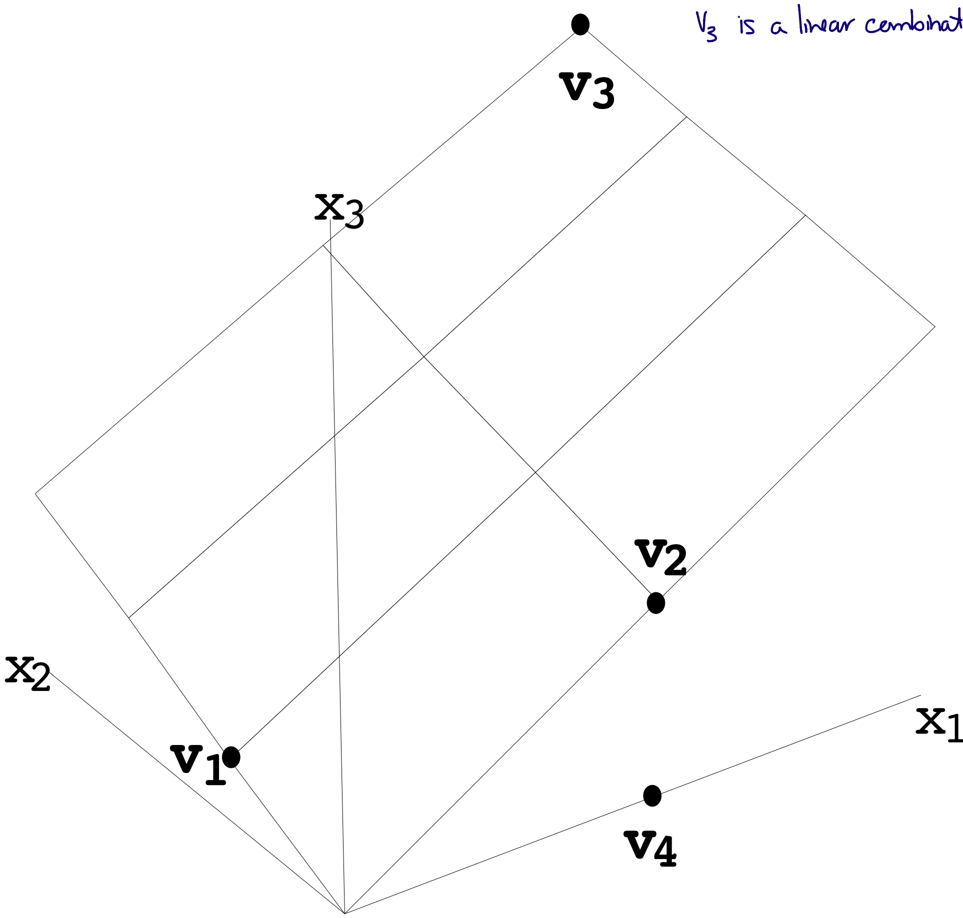
Independent,
b/c only one vector and is nonzero.

Characterization of Linearly Dependent Sets

EXAMPLE Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathbf{R}^3$ in the following diagram. Is the set linearly dependent? Explain

Yes, because

v_3 is a linear combination of v_1, v_2.



Theorem 7

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector $\mathbf{v}_j$ ($j \geq 2$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.