1.5 Solutions Sets of Linear Systems

Homogeneous System:

$$A\mathbf{x} = \mathbf{0}$$

(A is $m \times n$ and $\mathbf{0}$ is the zero vector in $\mathbf{R}^m$)

EXAMPLE:

$$\begin{aligned} x_1 &+ 10x_2 &= 0 \\ 2x_1 &+ 20x_2 &= 0 \end{aligned}$$

Corresponding matrix equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Trivial solution:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \mathbf{x} = \mathbf{0}$$

The homogeneous system $A\mathbf{x} = \mathbf{0}$ ***always*** has the **trivial solution**, $\mathbf{x} = \mathbf{0}$.

Nonzero vector solutions are called **nontrivial solutions.**

Do **nontrivial** solutions exist?

$$\begin{bmatrix} 1 & 10 & 0 \\ 2 & 20 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consistent system with a free variable has infinitely many solutions.

A homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions if

and only if the system of equations has

at least one free variable.

EXAMPLE: Determine if the following homogeneous system has nontrivial solutions and then describe the solution set.

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 0 \\ 4x_1 + 8x_2 - 10x_3 &= 0 \end{aligned}$$

Solution:

There is at least one free variable (why?)

$\Rightarrow$ nontrivial solutions exist

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

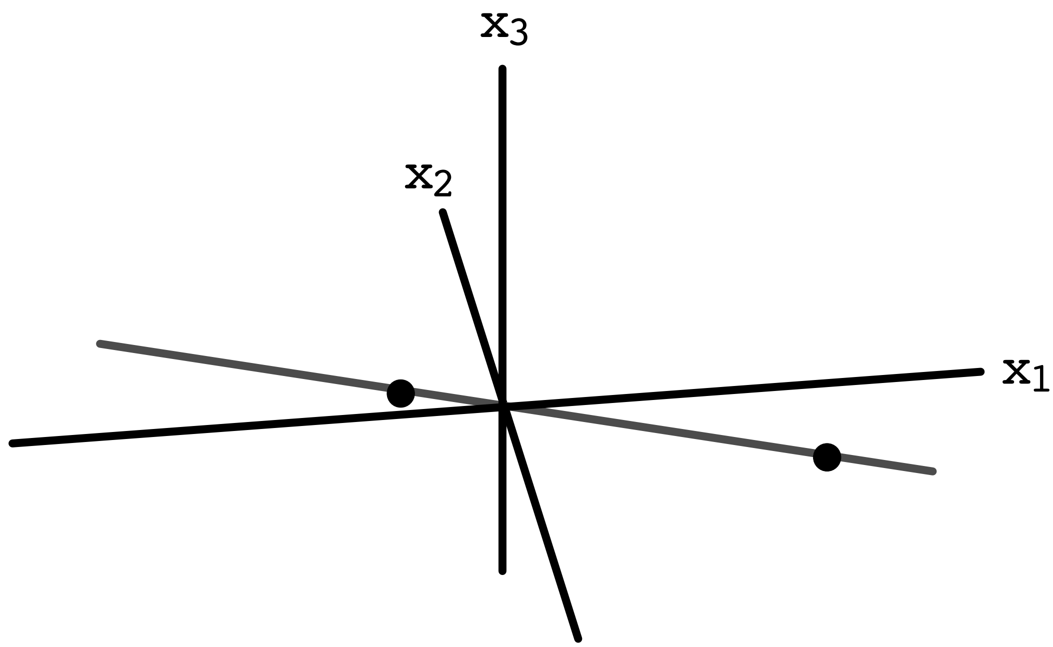
$x_1 = -2x_2$

x_2 is free

$x_3 = 0$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

Graphical representation:



solution set = span$\{\mathbf{v}\}$ = line through $\mathbf{0}$ in $\mathbf{R}^3$

EXAMPLE: Describe the solution set of

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 0 \\ 4x_1 + 8x_2 - 10x_3 &= 4 \end{aligned}$$

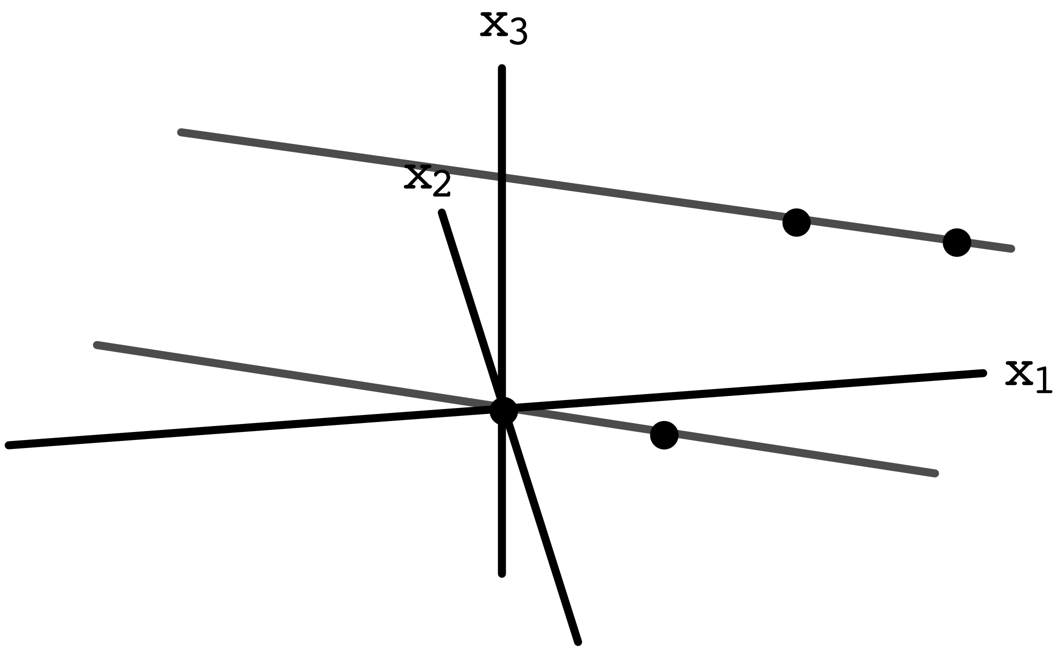
(same left side as in the previous example)

Solution:

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 \\ 0 + x_2 \\ 2 + 0x_2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}$$



Parallel solution sets of $A\mathbf{x} = \mathbf{0}$ & $A\mathbf{x} = \mathbf{b}$

Recap of Previous Two Examples

Solution of $A\mathbf{x} = \mathbf{0}$

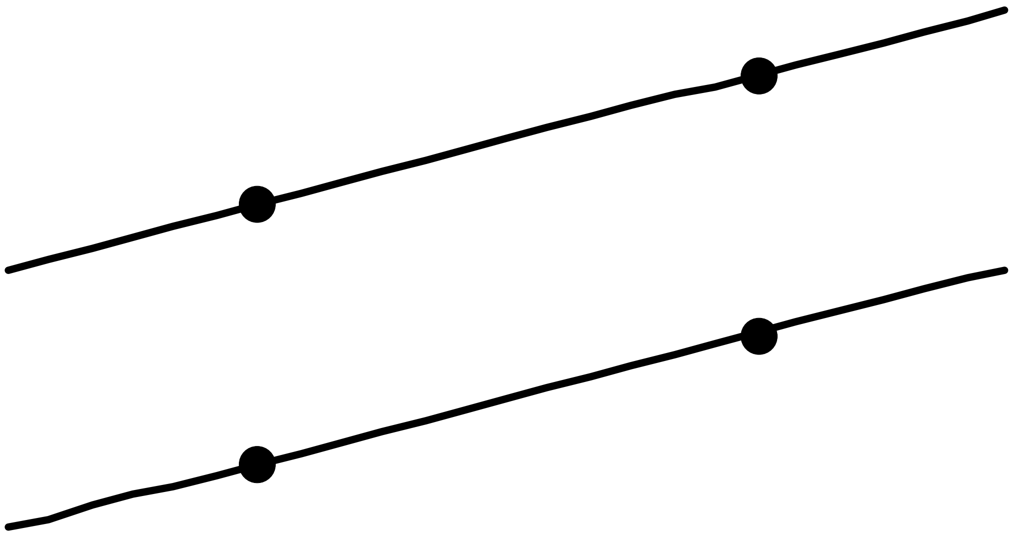
$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

$\mathbf{x} = x_2\mathbf{v}$ = parametric equation of line passing through **0** and **v**

Solution of $A\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2\mathbf{v}$$

$\mathbf{x} = \mathbf{p} + x_2\mathbf{v}$ = parametric equation of line passing through **p** parallel to **v**



Parallel solution sets of

$A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

EXAMPLE: Describe the solution set of $2x_1 - 4x_2 - 4x_3 = 0$; compare it to the solution set $2x_1 - 4x_2 - 4x_3 = 6$.

Solution: Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 0$:

$$\begin{bmatrix} 2 & -4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -2 & 0 \end{bmatrix}$$ (fill-in)

Vector form of the solution:

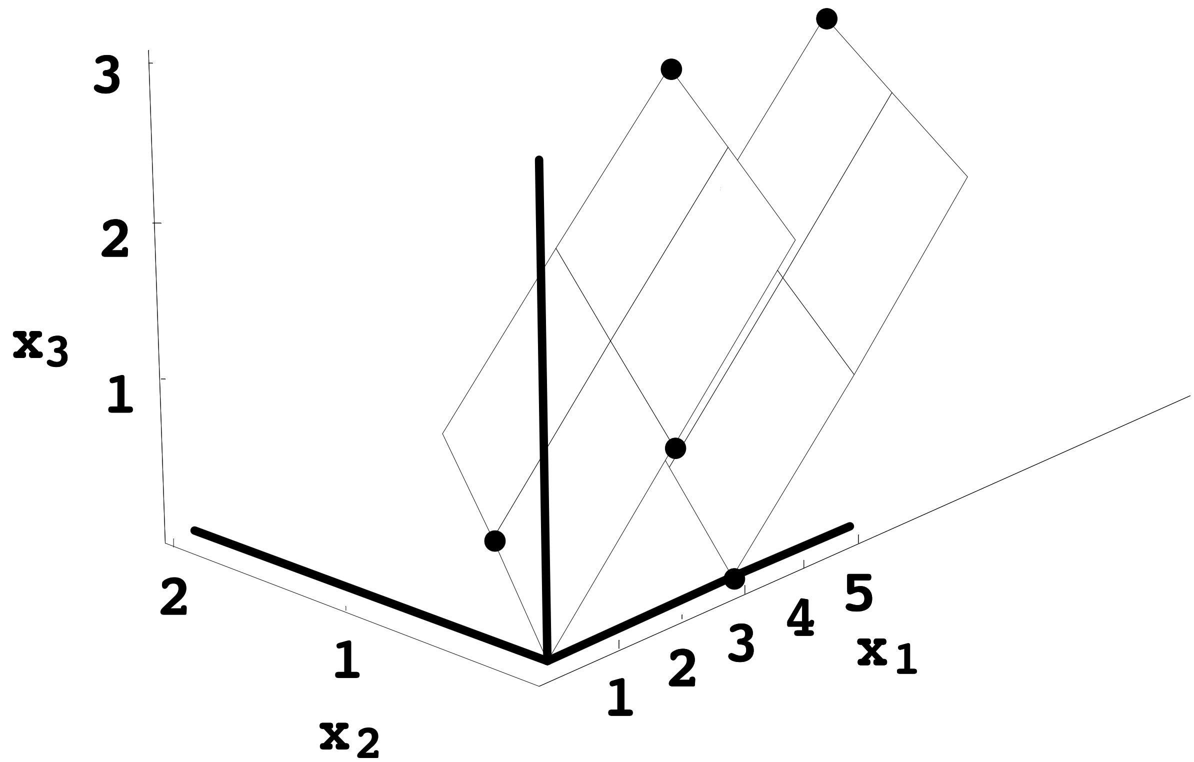
$$\mathbf{v} = \begin{bmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \underline{x_2}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \underline{x_3}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 6$:

$$\begin{bmatrix} 2 & -4 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -2 & 3 \end{bmatrix}$$ (fill -in)

Vector form of the solution:

$$\mathbf{v} = \begin{bmatrix} 3 + 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \underline{x_2}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \underline{x_3}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$



Parallel Solution Sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$