1.4 The Matrix Equation Ax = b

Linear combinations can be viewed as a matrix-vector multiplication.

Definition

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of A and x**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in x as weights**. I.e.,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

EXAMPLE:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6\begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

EXAMPLE: Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is a 3×1 vector.

$$\begin{bmatrix} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{bmatrix}$$

Solution: Corresponding system of equations (fill-in)

$$2x_1 + 3x_2 + 4x_3 = 9$$
$$-3x_1 + x_2 + 0x_3 = -2$$

Vector Equation:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Matrix equation (fill-in):

$$\begin{bmatrix} 2 & 3 & 4 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

Three equivalent ways of viewing a linear system:

1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$**; or**
3. as a matrix equation $A\mathbf{x} = \mathbf{b}$.

THEOREM 3

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbf{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Useful Fact:

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a

linear combination of the columns of A.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is __not__ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

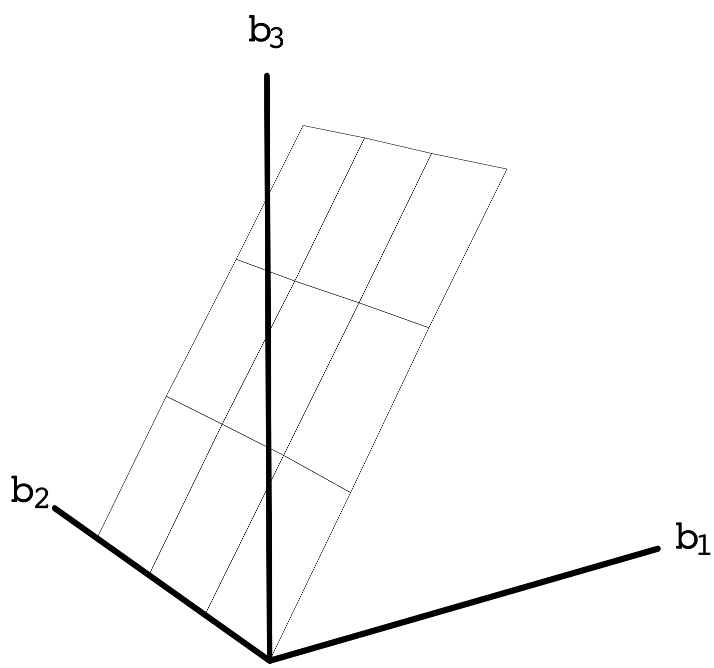
$$
\begin{array}{cc}
A = & \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix} \\
 & \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array}
\end{array}
$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)

$x_1\mathbf{a}_1 + x_2\mathbf{a}_3 + x_3\mathbf{a}_3 = \mathbf{b}$ if and only if $b_3 - 2b_1 = 0$.



Columns of A span a plane

in $\mathbf{R}^3$ through $\mathbf{0}$

Instead, if *any* **b** in $\mathbf{R}^3$ (not just those lying on a particular line or in a plane) can be expressed as a linear combination of the columns of A, then we say that the columns of A span $\mathbf{R}^3$.

Definition

We say that **the columns of** $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$ **span** $\mathbf{R}^m$ if ***every*** vector **b** in $\mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$

(i.e. Span$\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} = \mathbf{R}^m$).

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

a. For each **b** in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each **b** in $\mathbf{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbf{R}^m$.

d. A has a pivot position in every row.

Proof (outline): Statements (a), (b) and (c) are logically equivalent.

To complete the proof, we need to show that (a) is true when (d) is true and (a) is false when (d) is false.

Suppose (d) is true. Then row-reduce the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$$

and each row of U has a pivot position and so there is no pivot in the last column of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$.

So (a) is true.

Now suppose (d) is false. Then the last row of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ contains all zeros.

Suppose **d** is a vector with a 1 as the last entry. Then $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ represents an inconsistent system.

Row operations are reversible: $\begin{bmatrix} U & \mathbf{d} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is inconsistent also. So (a) is false. ■

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $\mathbf{b}$?

Solution: A has only __2__ columns and therefore has at most __2__ pivots.

Since A does not have a pivot in every __row__, $A\mathbf{x} = \mathbf{b}$ is __not__ __consistent__ for all possible $\mathbf{b}$,

according to Theorem 4.

EXAMPLE: Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbf{R}^3$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & 9 \end{bmatrix}$$

(no pivot in row 2)

By Theorem 4, the columns of A

do not span $\mathbb{R}^3$.

Another method for computing Ax

Read Example 4 on page 44 through Example 5 on page 45 to learn this rule for computing the product A**x**.

Theorem 5

If A is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbf{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = cA\mathbf{u}$.