MATH 61-02: WORKSHEET 6 (§4.4)

(W1) How many solutions does the equation $a + b + c + d + e = 2016$ have, if...

(a) a, b, c, d, e are all positive integers?

(b) a, b, c, d, e are all non-negative integers?

(c) a, b, c, d, e are all integers ≥ 10? (Hint: Let $A = a - 9, B = b - 9$, etc.)

(d) a, b, c, d, e are all even nonnegative integers?

(e) a, b, c, d, e are all integers (possibly negative) less than or equal to 2000? (Hint: here, $a \leq 2000$. Find a transformed variable $A \geq 1$ to set up stars-n-bars.)

Answer. (a) The answer is $\binom{2015}{4}$. Here's a good way to imagine this: Suppose we had 2016 objects laid in a row. We need to place four (indistinguishable) dividers in the gaps between the objects, such that the four dividers defined a division of our objects into five groups. Note that this doesn't allow us to put two dividers in the same spot. Then, since there are 2015 gaps between the objects and four dividers to place, there are $\binom{2015}{4}$ ways to do this, corresponding to solutions to the equation.

(b) This is a straightforward application of Theorem 4.4.8 in the book: $\binom{2020}{4}$.

(c) If we let $A = a-9, B = b-9, C = c-9, D = d-9, E = e-9$, then we have that A, B, C, D, E are all positive integers and we seek the number of solutions to the equation $a+b+c+d+e = 2016$, or $A+B+C+D+E = 1971$. By similar logic to part (b), the number of solutions this equation has in positive integers is $\binom{1970}{4}$.

(d) If a, b, c, d, e are even nonnegative integers, then $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{d}{2}, \frac{e}{2}$ are nonnegative and can take on any value in the nonnegative integers. Then $\frac{a}{2} + \frac{b}{2} + \frac{c}{2} + \frac{d}{2} + \frac{e}{2} = 1008$, and by Theorem 4.4.8 there are $\binom{1012}{4}$ solutions to this equation.

(e) Let $a' = 2000 - a, b' = 2000 - b, c' = 2000 - c, d' = 2000 - d, e' = 2000 - e$. Then a', b', c', d', e' are all nonnegative integers, and $a' + b' + c' + d' + e' = 5(2000) - (a + b + c + d + e) = 7984$. By Theorem 4.4.8 there are $\binom{7988}{4}$ solutions to this equation.

(W2) Recall that a *quadratic polynomial* in the variable x is an expression of the form $ax^2 + bx + c$. A *cubic polynomial* has degree three instead of two.

(a) What is the form of an arbitrary cubic polynomial in x? If your polynomial is called $g(x)$, evaluate $g(0), g(1), g(-1)$, and $g(2)$ in terms of the coefficients you used in your expression.

(b) How many cubic polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 9$?

(c) How many degree 6 polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 30$ and $f(-1) = 12$?

Answer. (a) An arbitrary cubic polynomial in x has form $a_3x^3 + a_2x^2 + a_1x^1 + a_0$. If this is $g(x)$, plugging in our values for x gives $g(0) = a_0$, $g(1) = a_3 + a_2 + a_1 + a_0$, $g(-1) = -a_3 + a_2 - a_1 + a_0$, and $g(2) = 8a_3 + 4a_2 + 2a_1 + a_0$.

(b) Let $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be an arbitrary cubic polynomial. Then we have that $f(1) = a_3 + a_2 + a_1 + a_0 = 9$, so we can count the number of such cubic polynomials by counting the number of solutions to this equation. By Theorem 4.4.8, there must be $\binom{12}{3}$ of these.

(c) Let $f(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be an arbitrary degree 6 polynomial. Then we have that

$$f(1) = a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = 30,$$

and

$$f(-1) = a_6 - a_5 + a_4 - a_3 + a_2 - a_1 + a_0 = 12.$$

If we add the two equations, we get that $a_6 + a_4 + a_2 + a_0 = 21$, and if we subtract the second equation from the first, we get that $a_5 + a_3 + a_1 = 9$. It is easy to see that any set of coefficients which satisfy these two equations will yield a desired degree 6 polynomial. Now, by stars-and-bars there are $\binom{20}{3}$ solutions to the first equation, and $\binom{8}{2}$ solutions to the second, so the number of combined solutions - hence solutions to the original problem - is $\binom{20}{3}\binom{8}{2}$.