# MATH 61-02: WORKSHEET 10 (§6.1-6.4)

(W1) Show that $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{5}}}}$ is an algebraic number.

Answer. Let $\alpha = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{5}}}}$. Then we have that

$$\alpha = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{5}}}} \implies \alpha^2 = 1+\sqrt{2+\sqrt{3+\sqrt{5}}} \implies \alpha^2 - 1 = \sqrt{2+\sqrt{3+\sqrt{5}}}$$
$$\implies (\alpha^2-1)^2 = 2+\sqrt{3+\sqrt{5}} \implies (\alpha^2-1)^2 - 2 = \sqrt{3+\sqrt{5}} \implies ((\alpha^2-1)^2-1)^2 = 3+\sqrt{5}$$
$$\implies ((\alpha^2-1)^2-1)^2 - 3 = \sqrt{5} \implies (((\alpha^2-1)^2-1)^2-3)^2 - 5 = 0,$$

and the left side gives a polynomial with integer coefficients $((((x^2-1)^2-1)^2-3)^2-5)$ with $\alpha$ as a solution, implying that $\alpha$ is an algebraic number, as desired. If you write out the polynomial, you get one of sixteenth-degree, namely $x^{16} - 8x^{14} + 24x^{12} - 32x^{10} + 10x^8 + 24x^6 - 24x^4 + 4$.

(W2) Prove that if the interval $[0,1]$ is partitioned into nondegenerate intervals (i.e., points don't count as intervals), then the partition is countable. Explain why this implies the same result for partitions of $\mathbb{R}$.

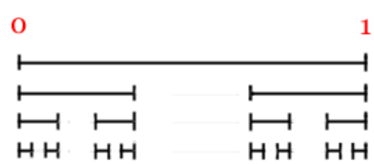
Hint: first list all the intervals of length $> 1/2$...

Answer. Fix any partition $\mathcal{P}$ of $[0,1]$. That partition must have only finitely many subintervals of length greater than $1/n$... certainly it can't have more than $n$ of them, because the pieces of a partition are disjoint. So let's list all the intervals within $\mathcal{P}$ in order of size, from biggest to smallest. If there's a tie, I'll list them from left to right.

For instance, if $\mathcal{P} = \{\ [0,.1),\ [.1,.4],\ (.4,.5],\ (.5,.75),\ [.75,1]\}$, then the size-ordered list would be $[.1,.4]$, $(.5,.75)$, $[.75,1]$, $[0,.1)$, $(.4,.5]$. Now I must see that this list will eventually exhaust the whole partition in the case that the partition is infinite. Consider a subinterval $I$ in $\mathcal{P}$. It has some positive length, so there is some $n$ such that $1/n < |I|$. But then there are no more than $n-1$ intervals that are longer than or equal to the length of $I$, so it comes no later than $n$th in the list! So I know everything eventually gets listed.

(W3) Show that the Cantor set is uncountable. (See Exercise 6.4.8 in the book.) This one is easily googled, but try it on your own!

Answer. Here is a simple solution. The Cantor set is created in stages; at each stage, you delete the middle third of all intervals.




So picking a point in the Cantor set is just picking whether to belong to the left-hand side or the right-hand side at each successive division of the intervals. That means that a point in the Cantor set corresponds to an infinite string of letters LLLRRLRLRLLL.... And we know from the previous worksheet (from a simple diagonalization argument) that this is uncountable.

**Alternate solution.** If you know what base 3 is, you can use that to solve this problem too:

Firstly, recall that in base 3 (or ternary) representation,

$$0.d_1d_2d_3... = 0 + d_1 3^{-1} + d_2 3^{-2} + d_3 3^{-3} + ...$$

where $d_i \in \{0, 1, 2\}, \forall i \in \mathbb{N}$. I claim that $C$ is the set of real numbers in $[0, 1]$ that can be represented in base 3 using only zeroes and twos: Suppose that a real number $x \in C$, but the $i$th digit after the decimal point of $x$ is a 1. We have three cases:

(1) If $x = 0.d_1d_2...d_{i-1}100000...$, then $x$ can be written as $x = 0.d_1d_2...d_{i-1}022222....$
(2) If $x = 0.d_1d_2...d_{i-1}122222...$, then $x$ can be written as $x = 0.d_1d_2...d_{i-1}200000....$
(3) If $x = 0.d_1d_2...d_{i-1}1d_{i+1}...$ where all digits after $d_i$ are not all zeros or all twos, then $x$ is in the open interval $(0.d_1d_2...d_{i-1}1, 0.d_1d_2...d_{i-1}2)$ and hence is a middle third that is removed in the construction of the Cantor set, so $x \notin C$.

Now, this implies that $C$ is uncountable: Suppose that $C$ were countable. Then $C$ can be expressed as a set $\{x_1, x_2, ...\} = \{x_i\}_{i\in\mathbb{N}}$. But now we can construct an element $x$ of $C$ that is not in this enumeration: Take the first digit after the decimal point of $x_1$, the second digit after the decimal point of $x_2$, and so on, and change every 0 to a 2 and every 2 to a 0. Since every $x_i$ can be represented in base 3 using only zeroes and twos, $x$ can as well, so it must be in $C$, but for every $i$ there is at least one digit of $x$ that does not match up with $x_i$, so $x$ cannot be any of the $x_i$'s, which is a contradiction. Hence $C$ is uncountable.