1\. a) $D\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $\vec{x} = c_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

2\. $(t^2-1)\dfrac{dx}{dt} = -x$ either $x \equiv 0$ or

$\dfrac{dx}{x} = \dfrac{-dt}{t^2-1} = \dfrac{1}{2}\left(\dfrac{1}{t+1} - \dfrac{1}{t-1}\right)$ so

$\ln|x| = \ln\sqrt{\dfrac{t+1}{t-1}} + C$ $\quad x = K\sqrt{\dfrac{t+1}{t-1}}$ K arbitrary.

$x(2) = K\sqrt{3} = 1$ $\quad K = 1/\sqrt{3}$ and $x = \sqrt{\dfrac{t+1}{3(t-1)}}$

3\. $\mathcal{L}x' = s\mathcal{L}x - 1$ gives

$(s-1)\mathcal{L}x - 1 = \dfrac{2}{s^3}$

$\mathcal{L}x = \dfrac{1}{s-1} + \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^3} + \dfrac{E}{s-1}$ Partial fractions gives

$A = -2 \quad B = -2 \quad C = -2 \quad E = 2$

So $x = e^t - 2 - 2t - t^2 + 2e^t = 3e^t - 2 - 2t - t^2$

4\. $\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\lambda = 1 \pm i$

With $\lambda = 1+i$ we get $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$

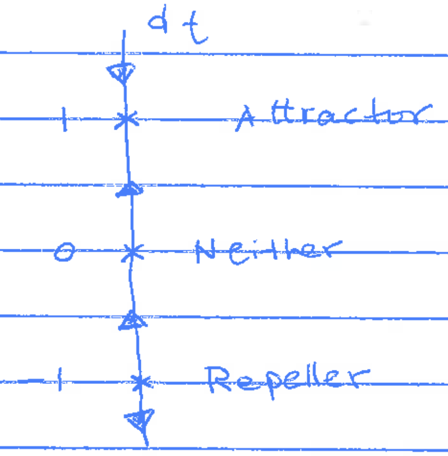
we get e-vector $\begin{pmatrix} i \\ 1 \end{pmatrix}$

Solutions: $e^t(\cos t + i\sin t)\begin{pmatrix} i \\ 1 \end{pmatrix} =$

$e^t\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i\, e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. This gives

general solution of $c_1 e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

5. $\frac{dx}{dt} = x^2(1-x)(1+x)$ gives equilibria at $0, 1, -1$



6 $D(D+1)(D-1)\,x = te^t$ Annihilator: $(D-1)^2$

New Eq. $D(D+1)(D-1)^3 x = 0$

Simplified guess $Ate^t + Bt^2e^t$

$D(D+1)(D-1)[Ate^t + Bt^2e^t] = e^t (D+1)(D+2)D[At + Bt^2]$

$= e^t (D+1)(D+2)[A + 2Bt] = e^t(2A + 6B + 4Bt)$

so $B = \frac{1}{4}$ $2A + 6B = 0$ gives $A = -\frac{3}{4}$

Par

Particular Solution $-\frac{3}{4}te^{t} + \frac{1}{4}t^2e^{t}$

General solution

$$c_1 + c_2e^{-t} + c_3e^{t} - \frac{3}{4}te^{t} + \frac{1}{4}t^2e^{t}$$

7) $\partial E/\partial x = 2xy - y^2 + y = g(x,y)$

$\partial E/\partial y = x^2 - 2xy + x = -f(x,y)$

$$\frac{\partial E}{\partial x}f(x,y) + \frac{\partial E}{\partial y}g(x,y) = 0$$

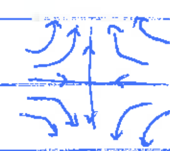
a) By above E is a constant of motion.

b) You can say E is also a Lyapunov function but if you did a) correctly you can also answer "no" for b)

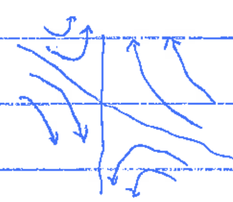
c) $(-1,0)$, $(0,1)$ and $\left(-\frac{1}{3}, \frac{1}{3}\right)$ and the origin

d) $A = \begin{pmatrix} 2y-2x-1 & 2x \\ 2y & 2x-2y+1 \end{pmatrix}$

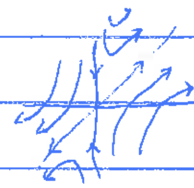
at $(0,0)$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ e-values $-1, 1$



at $(-1,0)$ $\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$ e-values 1 and -1



at $(0,1)$ $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ e-values $1, -1$



at $\left(-\frac{1}{3}, \frac{1}{3}\right)$ $\begin{pmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{pmatrix}$ e-values $\pm \frac{i}{\sqrt{3}}$

purely imaginary so HG does not work.

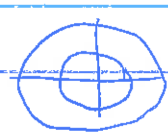
However you can use a): $\partial^2 E/\partial x^2 = 2y +$

$\partial^2 E/\partial x \partial y = 2x - 2y + 1$

$\partial^2 E/\partial y^2 = -2x$

gives $\Delta = \begin{vmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{vmatrix} = \frac{3}{3} > 0$ and the upper left corner is > 0

so we have a minimum of E at $\left(-\frac{1}{3}, \frac{1}{3}\right)$ and the phase portrait looks like



e) the first three are unstable and HG does not work for $\left(-\frac{1}{3}, \frac{1}{3}\right)$, but it is stable (by above)

f) the first three are neither and HG does not work for $\left(-\frac{1}{3}, \frac{1}{3}\right)$, but it is neither (by above)