Mathematics 38 | Tufts UNIVERSITY | **Differential Equations**

Final Exam | **May 9, 2011 8:30 - 10:30 AM**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and **cross out any work you do not want graded.** Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (30 points, 5 each) **No partial credit.**

a. Check for independence of:

1) the collection of functions $x,\ x\ln x,\ x\ln x^2$;

[dependent; $x\ln x^2 = 2x\ln x$, which is 2 times of the second function]

2) the collection of vectors

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}8\\7\\6\\5\end{pmatrix}, \begin{pmatrix}13\\8\\3\\-2\end{pmatrix}, \begin{pmatrix}12\\10\\4\\-1\end{pmatrix};$$

[dependent; 2 times of the second column minus 3 times of the first equals the third column, implying dependence]

b. Use the definition of the Laplace transform to compute $\mathscr{L}[te^{-t}]$;

[$\mathscr{L}[te^{-t}] = \int_0^\infty te^{-t}e^{-st}dt = \int_0^\infty te^{-(s+1)t}dt = \lim_{A\to\infty}\left(-\frac{1}{s+1}te^{-(s+1)t}|_0^A - \frac{1}{(s+1)^2}e^{-(s+1)t}|_0^A\right) = \frac{1}{(s+1)^2}$.]

c. Find $(D^2+2D+1)[e^{2t}\sin t]$;

[$(D^2+2D+1)[e^{2t}\sin t] = (D+1)^2[e^{2t}\sin t] = e^{2t}(D+3)^2[\sin t] = e^{2t}[(\sin t)'' + 6(\sin t)' + 9\sin t = e^{2t}(8\sin t + 6\cos t)$.]

d. Evaluate $e^t * t$;

[$e^t * t = \int_0^t e^{t-u}udu = e^t\left(\int_0^t e^{-u}udu\right) = e^t\left(-ue^u|_0^t - e^{-u}|_0^t\right) = e^t(-te^{-t} - e^{-t} + 1) = e^t - t - 1$.]

e. Find all solutions of the equation

$$x' + x = x^2;$$

[$x = 0$ and $x = 1$ by inspection. Find other possible solutions using separation:

$$\frac{dx}{x(x-1)} = dt,$$

or

$$\int\frac{dx}{x-1} - \int\frac{dx}{x} = \int dt,$$

or

$$\ln|\frac{x-1}{x}| = t + \ln C,$$

or

$$x(t) = (1 - Ce^t)^{-1}.$$

Notice, $x = 1$ if $C = 0$. Hence, answer is $x = 0$ and $x(t) = (1 - Ce^t)^{-1}$.]

f. Solve the initial value problem

$$x' + x = x^2, \qquad x(0) = \frac{1}{4}.$$

[From Part (e) $x(0) = \frac{1}{1-C} = 1/4$ implying $C = 3$.]

2. (6 points) Find the general solution of the non-homogeneous equation

$$x' + x\tan t = \sin 2t.$$

[First, the homogeneous equation has the general solution $x = c\cos t$. Variation of Parameters gives $p(t) = -2(\cos t)^2$. Hence, $x(t) = C\cos t - 2(\cos t)^2$.]

3. (10 points) Consider the following initial value problem:

$$x^{'} = \sqrt{|x|},\ x(0) = 0.$$

a. Is the existence and uniqueness theorem applicable?
[No, since $f(t,x) = \sqrt{|x|}$ is not differentiable at $x_0 = 0$;]

b. If it is not applicable, does the IVP has a solution?
[Yes, e.g. $x = 0$ by inspection;]

c. If a solution exists, is it unique? Explain.
[No, e.g. $x(t) = t^2/4$ (which can be aesily found by separation of variables) is also a solution to the given IVP.]

4. (8 points) Find the inverse Laplace transform of

a. $\dfrac{2s-1}{s^2-4s+8}$;
$[\dfrac{2s-1}{s^2-4s+8} = 2\dfrac{s-2}{(s-2)^2+4} + \dfrac{3}{2}\dfrac{2}{(s-2)^2+4}$, implying $\mathscr{L}^{-1}[\dfrac{2s-1}{s^2-4s+8}] = 2e^{2t}\cos 2t + \dfrac{3}{2}e^{2t}\sin 2t;]$

b. $\dfrac{e^{-s}s}{s^3+4s^2+4s}$.
$[\mathscr{L}^{-1}[\dfrac{e^{-s}s}{s^3+4s^2+4s}] = \mathscr{L}^{-1}[\dfrac{e^{-s}}{(s+2)^2}] = u_1(t)\mathscr{L}^{-1}[\dfrac{1}{(s+2)^2}](t-1) = u_1(t)e^{2-2t}(t-1).]$

5. (8 points) Let the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

be given.

a. A has an eigenvalue $\lambda = 1$. Find an eigenvector corresponding to this eigenvalue;
$[v = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}]$

b. The vector $v = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$ is an eigenvector of A. Find the corresponding eigenvalue.
$[\lambda = 1 - i.]$

6. (10 points) Use the Laplace transform method to solve

$$(D^3 - D)x = \begin{cases} 1, & if\ \ t < 2, \\ 0, & if\ \ t \geq 2, \end{cases} \qquad x(0) = x'(0) = x^{''}(0) = 0.$$

No credit for any other method.
[Rewrite the equation in the form $(D^3 - D)x = 1 - u_2(t)$ and then apply the Laplace transform. This gives

$$\mathscr{L}[x] = \frac{1}{s^2(s^2-1)} - \frac{e^{-2s}}{s^2(s^2-1)}.$$

Invert the first term

$$\mathscr{L}^{-1}[\frac{1}{s^2(s^2-1)}] = \mathscr{L}^{-1}\left[\frac{1}{2(s-1)} - \frac{1}{2(s+1) - \frac{1}{s^2}}\right] = \frac{1}{2}(e^t - e^{-t}) - t.$$

Use this and Second Shift formula to invert the second term, obtaining

$$\mathscr{L}^{-1}[\frac{e^{-2s}}{s^2(s^2-1)}] = u_2(t)\left(\frac{1}{2}(e^{t-2} - e^{-t+2}) - t + 2\right).$$

Therefore, $x(t) = \frac{1}{2}(e^t - e^{-t}) - t - \frac{u_2(t)}{2}\left(e^{t-2} - e^{-t+2} - 2t + 4\right)$.]

7. (10 points) Solve the system of differential equations

$$D\vec{x} = \begin{pmatrix} 1 & -1 & -4 \\ 2 & -2 & -2 \\ 1 & 0 & -4 \end{pmatrix} \vec{x}.$$

[The characteristic polynomial is

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -1 & -4 \\ 2 & -2-\lambda & -2 \\ 1 & 0 & -4-\lambda \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 1 & -4-\lambda \end{vmatrix} - (2+\lambda) \begin{vmatrix} 1-\lambda & -4 \\ 1 & -4-\lambda \end{vmatrix}$$

$$-2(\lambda+3) - \lambda(\lambda+2)(\lambda+3) = -(\lambda+3)[(\lambda+1)^2+1],$$

giving eigenvalues $\lambda = -3$ and $\lambda = -1 \pm i$. Eigenvectors corresponding to $\lambda = -3$ and $\lambda = -1 - i$ respectively are $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Hence, three linearly independent vector-solutions to the given system are

$$h_1(t) = \begin{pmatrix} e^{-3t} \\ 0 \\ e^{-3t} \end{pmatrix}, \; h_2(t) = e^{-t} \begin{pmatrix} 3\cos t - \sin t \\ 3\cos t + \sin t \\ \cos t \end{pmatrix}, \; h_3(t) = e^{-t} \begin{pmatrix} \cos t + 3\sin t \\ -\cos t + 3\sin t \\ \sin t \end{pmatrix}.$$

]

8. (8 points) Solve the system of differential equations

$$D\vec{x} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

[The homogeneous part has the general solution $H(t) = c_1 h_1(t) + c_2 h_2(t)$, where $h_1(t) = \begin{pmatrix} -e^t \\ e^t \end{pmatrix}$, $h_2(t) = \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}$. The variation of parameters method leads to the system

$$-e^t c_1' + e^{5t} c_2' = 2e^{-2t},$$

$$e^t c_1' + e^{5t} c_2' = e^{-2t}.$$

Solving this system we have $c_1' = -\frac{1}{2}e^{-3t}$ and $c_2' = \frac{3}{2}e^{-7t}$. Integrating, $c_1 = \frac{1}{6}e^{-3t}$ and $c_2 = -\frac{3}{14}e^{-7t}$. Hence, a particular solution is $p(t) = -\frac{1}{21}\begin{pmatrix} 8e^{-2t} \\ e^{-2t} \end{pmatrix}$.

Answer: $x(t) = c_1 \begin{pmatrix} -e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} - \frac{1}{21}\begin{pmatrix} 8e^{-2t} \\ e^{-2t} \end{pmatrix}$.]

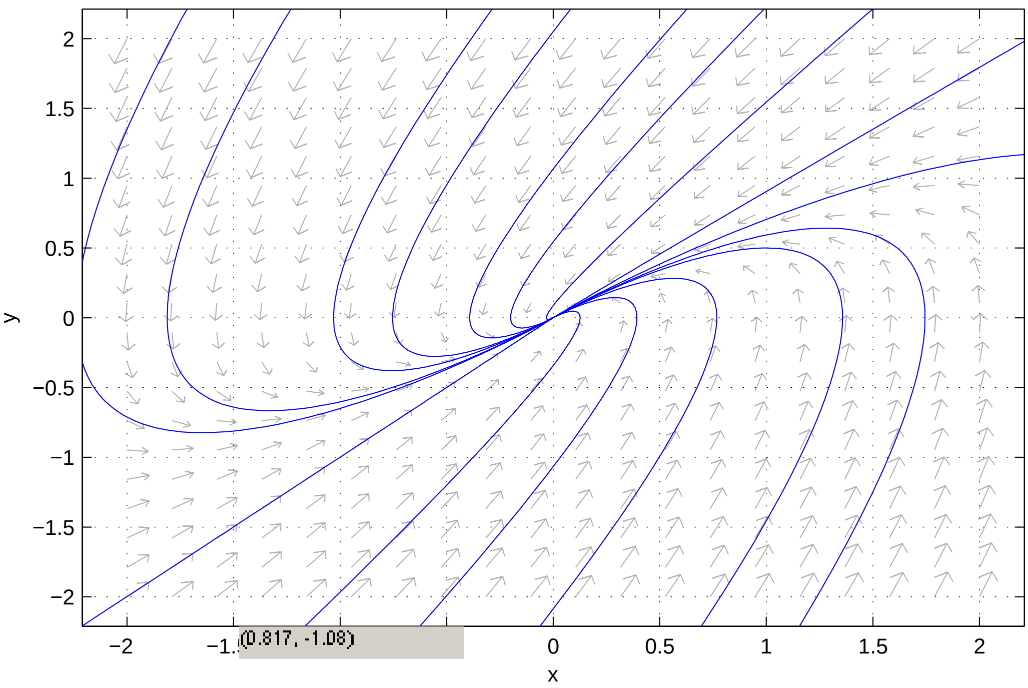
9. (10 points) Consider the system

$$\frac{dx}{dt} = -y,$$

$$\frac{dy}{dt} = x - 2y(1+x^2).$$

- **a.** Show that $E = x^2 + y^2$ is a Lyapunov function for this system;
  [$\frac{dE}{dt} = -xy + xy - 2y^2(1+x^2) \le 0$]
- **b.** Find equilibrium points;
  [The only equilibrium point is (0,0).]
- **c.** Classify each equilibrium;
  [$E(x,y) = x^2 + y^2$ has a local (even global) min at $(0,0)$. Therefore, the equilibrium point is a stable attractor;]
- **d.** Find the linearized matrix for each equilibrium point;
  [$A_{0,0} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$.]

e. Draw the phase portrait of the linearization of each equilibrium;



f. Determine whether the system has closed integral curves.
[Existence of the Lyapunov function excludes closed curves.]