Mathematics 51 Tufts UNIVERSITY **Differential Equations**
Examination II **April 8, 2013, 12–1:20**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. *Please box your answers and cross out any work you do not want graded.* Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (5 points) Determine whether the spring modeled by $mx'' + bx' + kx = 0$ with $m = 1\,\text{kg}$, $k = 3\,\text{N/m}$ and $b = 4\,\text{Ns/m}$ exhibits oscillations or not. (*No credit for just saying "yes" or "no."*)

Solution: No: The characteristic polynomial $1 \cdot r^2 + 4 \cdot r + 3$ has real roots.

2. (15 points) Find the terms up to t^3 in a power series expansion for a solution of

$$x''' - 3x'' + 3x' - x = 0, \qquad x(0) = 0,\ x'(0) = 0,\ x''(0) = 2.$$

Solution: Write $x(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \dots$ and note that $b_0 = b_1 = 0$ and $b_2 = 1$ from the initial conditions, so $x(t) = t^2 + b_3 t^3 + \dots$. Now, use the differential equation to either get

$$6b_3 = x'''(0) = 3x''(0) - 3x'(0) + x(0) = 3 \cdot 2 - 3 \cdot 0 + 0 = 6,$$

or to write $(6b_3 + \dots) - 3(2 + 6b_3 t + \dots) + 3(2t + 3b_3 t^2 + \dots) - (t^2 + b_3 t^3 + \dots) = 0$, which for $t = 0$ or by comparing constant terms gives $6b_3 - 6 = 0$. Either way, $b_3 = 1$, and $x(t) = t^2 + t^3 + \dots$

[One can check this by noting that the solution is $x(t) = t^2 e^t = t^2(1 + t + \dots)$.]

3. (10 points) Solve the initial-value problem $D\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Solution: Inspection gives eigenvalue 1 with eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and eigenvalue 0 (double) with eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so the general solution is $c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The initial condition gives $c_1 = 1$, $c_2 = 1$, $c_3 = 0$, so the solution is $\vec{x}(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

4. (15 points) Solve $D\vec{x} = A\vec{x}$, where $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

Solution: Note the block form. This is a decoupled pair of second-order systems of differential equations. The characteristic polynomial of $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ is $(1 - \lambda)(-1 - \lambda) - 3 = \lambda^2 - 4$, so the eigenvalues are ± 2. 2 has eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by inspection, for $\lambda = -2$ solve $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \vec{v} = 0$

to find $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Next, $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$, and solving the second equation of $(A - iI)\vec{v} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \vec{v} = \vec{0}$ gives eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ with associated solution

$$e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} = (\cos t + i \sin t) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i\cos t + \sin t \\ \cos t + i \sin t \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$

Thus, the general solution is $c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ \sin t \\ \cos t \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ -\cos t \\ \sin t \end{pmatrix}$.

5. (5 points) Find **one** nonzero solution of $D\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix} \vec{x}$.

(Yes, just one solution is good enough, so long as it is not the zero function.)

Solution: By inspection, 6 is an eigenvalue with eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and a corresponding solution is $\vec{x}(t) = e^{6t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

6. (15 points) Solve

$$(D-1)x = \begin{cases} 0 & t < 2 \\ 3 & t \geq 2 \end{cases}, \qquad x(0) = 1.$$

Solution: $(s-1)\mathscr{L}[x] - 1 = \mathscr{L}[3u_2(t)] = 3e^{-2s}\mathscr{L}[1] = \dfrac{3e^{-2s}}{s}$, so $\mathscr{L}[x] = \dfrac{1}{s-1} + \dfrac{3e^{-2s}}{s(s-1)}$.

Now do a partial fractions decomposition: $\dfrac{1}{s(s-1)} = \dfrac{s-(s-1)}{s(s-1)} = \dfrac{1}{s-1} - \dfrac{1}{s}$, so

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{3e^{-2s}}{s(s-1)}\right] = e^t + 3u_2(t)\mathscr{L}^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right](t-2) \\ &= e^t + 3u_2(t)\left(e^{t-2} - 1\right) = e^t - 3u_2(t)\left(1 - e^t/e^2\right) = \begin{cases} e^t & \text{for } t < 2 \\ (1+3e^{-2})e^t - 3 & \text{for } t \geq 2 \end{cases}. \end{aligned}$$

[To check, note that for $t < 2$ this is $(D-1)x = 0$ with $x(0) = 1$, giving e^t, and for $t \geq 2$ this is $(D-1)x = 3$ with $x(2) = e^2$, giving $ce^t - 3$ with $ce^2 - 3 = e^2$, i.e., $c = e^{-2}(e^2+3) = 1 + 3e^{-2}$.]

7. (5 points) Check the vectors $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ for linear independence.

Solution: Linearly dependent: 4 vectors of size 3.

8. (5 points, *no credit unless every answer is correct*) For each of the following vectors decide whether it is an eigenvector of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$, and if so, provide the corresponding eigenvalue. For each part, your answer should be either "NO" or a number; please put all your answers on the inside front blue cover of your examination booklet.

a. $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ **b.** $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ **c.** $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ **d.** $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ **e.** $\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ **f.** $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ **g.** $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

h. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **i.** $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ **j.** $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **k.** $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ **l.** $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ **m.** $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ **n.** $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Solution: Clearly, the eigenvalues are 1, 5, and 9 with eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively. Since we have 3 distinct eigenvalues and corresponding eigenvectors, for each part we only need to check whether the vector is a multiple of one of the 3 we already have; the answer is "No" in all cases (none of them contains 2 zeros).

9. (5 points) Determine whether

$$\begin{aligned} x_1(t) &= 3c_1e^{4t} + c_2e^{-4t} \\ x_2(t) &= c_1e^{4t} + c_2e^{-4t} \end{aligned}$$

describes the general solution of the system

$$\begin{aligned} x_1' &= 5x_1 - 3x_2, \\ x_2' &= 3x_1 - 5x_2. \end{aligned}$$

Solution: No, it is not even a solution unless $c_2 = 0$.

10. (5 points, no partial credit) The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has a triple eigenvalue 1. Find 3 linearly independent generalized eigenvectors (you do not have to verify that they are linearly independent).

Solution: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ will work because there must be 3 linearly independent ones.

END OF EXAMINATION