Mathematics 51 **Differential Equations**
Examination II **October 29, 2012, 12–1:20**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (5 points) Determine whether the spring modeled by $(mD^2 + bD + k)x = 0$ with $m = 1\,\text{kg}$, $k = 5\,\text{N/m}$ and $b = 4\,\text{Ns/m}$ exhibits oscillations or not. (*No credit for just saying "yes" or "no."*)

Solution: Yes: The characteristic polynomial $1 \cdot r^2 + 4 \cdot r + 5 = (r+2)^2 + 1$ has complex roots.

2. (10 points) Find the general solution of $x^{(4)} - 2x^{(3)} + x^{(2)} = 0$.

Solution: (Rewrite as $(D-1)^2 D^2 x = 0$ if desired.) $c_1 e^t + c_2 t e^t + c_3 + c_4 t$.

3. (15 points) Solve the initial-value problem $9x'' + x = 0$ with $x(0) = 2$ and $x'(0) = 3$.

Solution: The general solution is $c_1 \cos t/3 + c_2 \sin t/3$; the initial conditions give $c_1 = 2$ and $c_2 = 9$, so the desired solution is $x(t) = 2\cos t/3 + 9 \sin t/3$.

4. (15 points) Find the general solution of $(D-1)^4 x = 2t + e^{-t}$.

Solution: A simplified guess for a particular solution is $p(t) = c_1 + c_2 t + c_3 e^{-t}$; plug this in:

$$(D-1)^4(c_1 + c_2 t) = (\cdots - 4D + 1)(c_1 + c_2 t) = -4c_2 + c_1 + c_2 t$$

and, using the exponential shift,

$$(D-1)^4 c_3 e^{-t} = c_3 e^t D^4 e^{-2t} = c_3 e^t \cdot (-2)^4 e^{-2t} = c_3 e^t \cdot 16 e^{-2t} = 16 c_3 e^{-t},$$

so $2t + e^{-t} \stackrel{!}{=} (D-1)^4 p(t) = -4c_2 + c_1 + c_2 t + 16 c_3 e^{-t}$ and $c_3 = 1/16$, $c_2 = 2$, $c_1 = 8$. The general solution is $x(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + c_4 t^3 e^t + 8 + 2t + e^{-t}/16$.

5. (15 points) Find the general solution of

$$x'' - 2x' + x - \frac{e^t}{t^2} = 0.$$

(*Check all your intermediate answers carefully; no credit for work based on wrong prior steps.*)

Solution: The general solution of the associated homogeneous differential equation is $c_1 e^t + c_2 t e^t$; variation of parameters gives the solution $p(t) = c_1(t) e^t + c_2(t) t e^t$ with

$$\begin{aligned} c_1'(t)e^t + c_2'(t)te^t &= 0 \\ c_1'(t)e^t + c_2'(t)(t+1)e^t &= e^t/t^2, \end{aligned} \quad \text{or} \quad \begin{aligned} c_1'(t) + c_2'(t)t &= 0 \\ c_1'(t) + c_2'(t)(t+1) &= 1/t^2, \end{aligned} \quad \text{or} \quad \begin{aligned} c_1'(t) + c_2'(t)t &= 0 \\ c_2'(t) &= 1/t^2, \end{aligned}$$

so $c_2'(t) = 1/t^2$, $c_1'(t) = -1/t$, hence $c_1(t) = -1/t$, $c_2(t) = -\ln t$, and $p(t) = -e^t \ln t - \dfrac{1}{t} t e^t$. The general solution then is

$$x(t) = c_1 e^t + c_2 t e^t - e^t \ln t.$$

6. (15 points) Solve

$$(D-1)x = \begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases}, \qquad x(0) = 1.$$

Solution: $(s-1)\mathscr{L}[x] - 1 = \mathscr{L}[u_2(t)] = e^{-2s}\mathscr{L}[1] = \dfrac{e^{-2s}}{s}$, so $\mathscr{L}[x] = \dfrac{1}{s-1} + e^{-2s}\dfrac{1}{s(s-1)}$.

Now do a partial fractions decomposition: $\dfrac{1}{s(s-1)} = \dfrac{s-(s-1)}{s(s-1)} = \dfrac{1}{s-1} - \dfrac{1}{s}$, so

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{e^{-2s}}{s(s-1)}\right] = e^t + u_2(t)\mathscr{L}^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right](t-2)$$
$$= e^t + u_2(t)\left(e^{t-2} - 1\right) = e^t - u_2(t)\left(1 - e^t/e^2\right) = \begin{cases} e^t & \text{for } t < 2 \\ (1+e^{-2})e^t - 1 & \text{for } t \geq 2 \end{cases}.$$

[To check, note that for $t < 2$ this is $(D-1)x = 0$ with $x(0) = 1$, giving e^t, and for $t \geq 2$ this is $(D-1)x = 1$ with $x(2) = e^2$, giving $ce^t - 1$ with $ce^2 - 1 = e^2$, i.e., $c = e^{-2}(e^2+1) = 1 + e^{-2}$.]

7. (15 points) Find the terms up to t^3 in a power series expansion for a solution of

$$x''' - 3x'' + 3x' - x = 0, \qquad x(0) = 0,\ x'(0) = 0,\ x''(0) = 2.$$

Solution: Write $x(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \ldots$ and note that $b_0 = b_1 = 0$ and $b_2 = 1$ from the initial conditions, so $x(t) = t^2 + b_3 t^3 + \ldots$. Now, either use the differential equation to get

$$6b_3 = x'''(0) = 3x''(0) - 3x'(0) + x(0) = 3 \cdot 2 - 3 \cdot 0 + 0 = 6,$$

or to write $(6b_3 + \ldots) - 3(2 + 6b_3 t + \ldots) + 3(2t + 3b_3 t^2 + \ldots) - (t^2 + b_3 t^3 + \ldots) = 0$, which for $t = 0$ or by comparing constant terms gives $6b_3 - 6 = 0$. Either way, $b_3 = 1$, and $x(t) = t^2 + t^3 + \ldots$

[One can check this by noting that the solution is $x(t) = t^2 e^t = t^2(1 + t + \ldots)$.]

8. (10 points) Given the differential equation

$$(D^2 + 4D + 3)x = 3t + 7 \qquad \text{(N)}$$

- **a.** find the equivalent system $(\mathrm{S_N})$,
- **b.** the general solution of (N) is $x(t) = c_1 e^{-t} + c_2 e^{-3t} + t + 1$. *You do not need to verify this.* Use the general solution of (N) to obtain each component of the general solution of $(\mathrm{S_N})$,
- **c.** write $(\mathrm{S_N})$ in matrix form,
- **d.** write the general solution of $(\mathrm{S_N})$ in the form $\vec{x} = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t) + \vec{p}(t)$.

Solution: **a.** $\begin{cases} x_1' = x_2, \\ x_2' = -3x_1 - 4x_2 + 3t + 7. \end{cases}$ **b.** $\begin{cases} x_1(t) = c_1 e^{-t} + c_2 e^{-3t} + t + 1, \\ x_2(t) = -c_1 e^{-t} - 3c_2 e^{-3t} + 1. \end{cases}$

c. $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 3t+7 \end{pmatrix}$. **d.** $\vec{x} = c_1\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{-3t} \\ -3e^{-3t} \end{pmatrix} + \begin{pmatrix} t+1 \\ 1 \end{pmatrix}$.

END OF EXAMINATION