Mathematics 51 **Differential Equations**
Examination I **October 1, 2012, 12–1:20**

Tufts UNIVERSITY

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

Please write the solutions to problems 1–6 on the inside (blue!) front cover of your exam book.

1. (3 points each, no partial credit) For each of the differential equations below determine the order, determine whether the differential equation is linear, and if so, whether it is homogeneous.

a. $t^4\dfrac{d^3x}{dt^3} + t\dfrac{dx}{dt} - x - t^7 = 0$

b. $x^8\dfrac{dx}{dt} + \dfrac{d^7x}{dt^7} = x + t^9$

c. $\left(\dfrac{dx}{dt}\right)^5 + \dfrac{d^4x}{dt^4} - t^3x^7 + t^7 = 0$

d. $(x')^2x''' = x^4x'' + t^5x'$

Solution: **a.** 3, l, n, **b.** 7, n, **c.** 4, n, **d.** 3, n.

2. (3 points each, no partial credit) Find all real values of α for which the given function is a solution of the given differential equation.

a. $x = \alpha, \qquad (D^7 + 3D^6 - 2tD^5 + \pi D^4 + D^3 - D^2 + D - 1)x = 7$

b. $x = t^\alpha, t > 0, \qquad 16t^2x'' + 3x = 0$

c. $x = e^{\alpha t}, \qquad x'\sqrt{x} = 2e^{3t}$

Solution: **a.** -7, **b.** 1/4, 3/4, **c.** 2.

3. (2 points each, no partial credit) For each of the following differential equations state whether it is normal on $0 < t < 2$.

a. $(t-1)\dfrac{dx}{dt} - 5x = 3t$

b. $t\dfrac{dx}{dt} + e^tx = \sin t$

Solution: **a.** no, **b.** yes.

4. (5 points, no partial credit) For the initial-value problem $x' = x^2, \quad x(0) = -1$ use 2 steps of Euler's method to approximate $x(1)$.

Solution: $x(1/2) \approx -1 + \frac{1}{2}\cdot(-1)^2 = -1/2$ and $x(1) \approx -1/2 + \frac{1}{2}\cdot(-1/2)^2 = -3/8$.

5. (5 points, no partial credit) Evaluate the determinant $\det\begin{pmatrix} 0 & 1 & 6 & 9 & 1 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution: $5\cdot4\cdot3\cdot2\cdot1 = 120$.

6. (5 points, no partial credit) Compute the Wronskian of $h_1(t) = \sin t$ and $h_2(t) = t\cos t$ at $t = 0$.

Solution: $W[\sin t, t\cos t](0) = \begin{vmatrix} 0 & 0 \\ * & * \end{vmatrix} = 0$.

7. (10 points) Solve the initial-value problem $x\dfrac{dx}{dt} = -t^2, \quad x(0) = -5.$

Solution: Separation of variables: $x^2/2 = \int x\,dx = -\int t^2\,dt = -t^3/3 + C$. Insert $t = 0$, $x = -5$ to get $25 = (-5)^2 = 2C$, so $x = -\sqrt{25 - 2t^3/3}$.

8. (15 points) Solve the initial-value problem $\dfrac{dx}{dt} - tx = t, \quad x(0) = \dfrac{1}{2}.$

Solution: Rewrite as $x' = (x+1)t$ and separate variables: $\ln|x+1| = t^2/2 + c$, so $x + 1 = ke^{t^2/2}$, and $x(t) = ke^{t^2/2} - 1$. Now, $1/2 = x(0) = k - 1$, so $k = 3/2$, and $x(t) = \frac{3}{2}e^{t^2/2} - 1$.

Alternate solution: Take the associated homogenenous equation, $x' - tx = 0$ and separate variables to get $\ln|x| = t^2/2 + c$, so $x = ke^{t^2/2}$. By inspection $p(t) = -1$ is a particular solution or use the variation of parameters ansatz $x = k(t)e^{t^2/2}$ to get $k'(t)e^{t^2/2} = t$, or $k'(t) = te^{-t^2/2}$. Integrate by substituting $u = t^2/2$, so $du = tdt$, to find $k(t) = -e^{-t^2/2} + c$, hence

$$x(t) = \left(-e^{-t^2/2} + c\right)e^{t^2/2} = ce^{t^2/2} - 1.$$

Plugging in the initial condition gives $c = 3/2$, as in the other solution.

9. (5 points) Consider the functions t^5 and $|t^5|$ on $-\infty < t < \infty$. Are they linearly independent? Explain!

(Hint: Computing the Wronskian may *not* be the best approach.)

Solution: If $c_1t^5 + c_2|t^5| = 0$ for all t, then we can insert $t = 1$ to get $c_1 + c_2 = 0$ and $t = -1$ to get $-c_1 + c_2 = 0$; together these give $c_1 = 0$ and $c_2 = 0$, so these functions are linearly independent.

10. (10 points)

a. Check whether e^t and e^{-t} are solutions of $x'' - x = 0$.

b. Check whether e^t and e^{-t} generate the general solution of $x'' - x = 0$. (Give reasons!)

c. Find a constant solution of $x'' - x = 3$.

d. Find the general solution of $x'' - x = 3$.

Solution: **a.** Yes: $D^2e^{\pm t} - e^{\pm t} = e^{\pm t} - e^{\pm t} = 0$. **b.** Yes: $W[e^t, e^{-t}](0) = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0$. **c.** If x is constant, then $x'' = 0$, so $-x = 3$, and $x = -3$. **d.** $c_1e^t + c_2e^t - 3$.

11. (10 points) Determine whether the system

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + z &= 0 \\ y + z &= 0 \end{aligned}$$

has a unique solution or no solution or infinitely many solutions; find all solutions.

Solution: By Cramer's rule there is a unique solution, and it therefore (clearly) is $x = y = z = 0$.

12. (10 points) Find all solutions of the system

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 - x_5 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 1 \\ -x_1 \qquad\quad - x_3 + 2x_4 + x_5 &= 1 \end{aligned}$$

Solution: $\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & -1 & -1 & 1 \\ 2 & 2 & 2 & -3 & -2 & 1 \\ -1 & & -1 & 2 & 1 & 1 \end{array}\right) \xrightarrow{R3\to R3+R2-R1} \left(\begin{array}{rrrrr|r} 1 & 2 & 1 & -1 & -1 & 1 \\ 2 & 2 & 2 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$; there is no solution.

END OF EXAMINATION