Mathematics 51 | **Differential Equations**
Examination IV | **Dec. 14, 2012, 8:30–10:30am**

Tufts UNIVERSITY

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. *Please box your answers and cross out any work you do not want graded.* Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (5 points, no partial credit) Compute $x(2)$, where $x(t)$ is the solution of $tx' = 2x$ with $x(1) = 3$.

2. (5 points) Determine whether

$$\begin{aligned} x_1(t) &= 3c_1e^{4t} + c_2e^{-4t} \\ x_2(t) &= \ \ c_1e^{4t} + c_2e^{-4t} \end{aligned}$$

describes the general solution of the system

$$\begin{aligned} x_1' &= 5x_1 - 3x_2, \\ x_2' &= 3x_1 - 5x_2. \end{aligned}$$

3. (5 points, no partial credit) The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has a triple eigenvalue 1. Find 3 linearly independent generalized eigenvectors (you do not have to verify that they are linearly independent).

4. (10 points) Solve the initial-value problem $D\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$

5. (10 points) Find the general solution of $x'' + x = \sec t$.

6. (15 points, limited partial credit) Find the general solution of $D\vec{x} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix} \vec{x}.$

You may use without checking that $\begin{pmatrix} 1 + \frac{t^2}{2} \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix}$ *are linearly independent solutions.*

7. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.
(No credit for answers with over 15 words.)

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8. (15 points) Consider the system
$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = e^x y - x.$$

a. Find all equilibria.

b. Draw the phase portrait of the *linearization* of each equilibrium.

c. For each equilibrium determine whether the Hartman–Grobman Theorem applies.

d. Decide whether $E(x,y) = -x^2 - y^2$ is a constant of motion.

e. Decide whether $E(x,y) = -x^2 - y^2$ is a Lyapunov function.

f. Classify each equilibrium as an attractor, a repeller, or neither of these.

g. Determine the stability of each equilibrium.

h. Decide whether this system of differential equations has a closed integral curve.

9. (10 points) Solve $(D^3 - D)x = \begin{cases} 1 & t < 2 \\ 0 & t \geq 2 \end{cases}, \qquad x(0) = x'(0) = x''(0) = 0.$

10. (10 points) Consider the differential equation $x'' + tx' + x = 2t$.

a. Find the power-series expansion of the solution with $x(0) = 0$, $x'(0) = 1$.

b. Find the equivalent system of differential equations.

c. Find the solution of that system for which $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

11. (5 points, *no credit unless every answer is correct*) For each of the following vectors decide whether it is an eigenvector of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and if so, provide the corresponding eigenvalue. For each part, your answer should be either "NO" or a number; please put all your answers on the inside front blue cover of your examination booklet.

a. $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ **b.** $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ **c.** $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ **d.** $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ **e.** $\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ **f.** $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ **g.** $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

h. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **i.** $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ **j.** $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **k.** $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ **l.** $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ **m.** $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ **n.** $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

END OF EXAMINATION