

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (15 points) Find the general solution of $\begin{cases} x' = 2y \\ y' = 3y - x \end{cases}$.
2. (15 points) Find the general solution of $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}$. [Hint: $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector.]
3. (10 points) The matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ has a double eigenvalue 2 with generalized eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the general solution of $D\vec{x} = A\vec{x}$.
4. (15 points) Find the general solution of $D\vec{x} = A\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$.
Hint: The general solution of $D\vec{x} = A\vec{x}$ is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
5. (15 points) Sketch the phase portrait of the system $D\vec{x} = A\vec{x}$, where A is the matrix from problem 4.

6. (20 points) For the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= -2x + x^2 \end{aligned}$$
 - a. find all equilibria, and for each of them
 - b. determine whether the Hartman–Grobman Theorem applies to it, and if so
 - c. determine its stability,
 - d. decide whether it is an attractor, a repeller, or neither,
 - e. draw a phase portrait of the linearization.
Sketch the full phase portrait.

7. (10 points) For the system

$$\begin{aligned} \frac{dx}{dt} &= -\sin x \cos y \\ \frac{dy}{dt} &= \cos x \sin y, \end{aligned}$$
 - a. check whether $E(x, y) = \sin x \sin y$ is a constant of motion,
 - b. find one equilibrium (yes, just one—but make sure you get it right: there will be no credit for any work on a point that is not an equilibrium, and no credit if you work on more than one point!),
 - c. determine whether it is an attractor, a repeller, or neither. (Give clear and complete reasoning!)