1.)

root	multiplicity	fxns
1	2	e^t, te^t
2	2	e^{2t}, te^{2t}
-2	3	$e^{-2t}, te^{-2t}, t^2e^{-2t}$

$$P(r) = (r-1)^2(r+2)(r^2-4)^2$$

$$= (r-1)^2(r+2)\left((r-2)(r+2)\right)^2$$

$$= (r-1)^2(r-2)^2(r+2)^3$$

gen'l sol'n:

$$x(t) = c_1e^t + c_2te^t + c_3e^{2t} + c_4te^{2t} + c_5e^{-2t} + c_6te^{-2t} + c_7t^2e^{-2t}$$

2) $\frac{dx}{dt} - \frac{1}{2t}x = \frac{1}{2}$ $\quad t>0 \quad$ (N)

(a) $\frac{dx}{dt} - \frac{1}{2t}x = 0$ $\quad$ (H)

(H) solved using SOV: $\frac{dx}{x} = \frac{1}{2t}dt$ $\quad$ assuming $x \neq 0$

integrating...

$$\ln|x| = \frac{1}{2}\ln|t| + c \overset{\text{since } t>0}{=} \frac{1}{2}\ln(t) + c$$

$$= \ln(t^{1/2}) + c$$

$$\Rightarrow x = c\sqrt{t}$$

Notice $x=0$ can be achieved w/ $c=0$

Try guess $x(t) = k(t)\sqrt{t}$

$$\Rightarrow x'(t) = k'\sqrt{t} + \frac{1}{2\sqrt{t}}k(t)$$

Then $x' - \frac{1}{2t}x = k'\sqrt{t} + \frac{1}{2\sqrt{t}}k(t) - \frac{1}{2t}k(t)\sqrt{t}$

$$= k'\sqrt{t}$$

Using (N) $\stackrel{(N)}{=} \frac{1}{2}$

So $k'(t) = \frac{1}{2\sqrt{t}}$ then $k(t) = \sqrt{t}$ So particular sol'n is $\sqrt{t}\cdot\sqrt{t} = t$

Gen'l sol'n: $x(t) = x_h(t) + x_p(t)$

$$= c\sqrt{t} + t$$

(b) Given $x(1) = 0$. Notice $x(1) = c+1$

So $c=-1$, and specific solution is

$$x(t) = t - \sqrt{t}$$

3) $\dfrac{dx}{dt} = x + x^{1/3}$; $x(0) = 0$

(a) Let $F(t,x) = x + x^{1/3}$

then $f_x(t,x) = 1 + \dfrac{1}{3x^{2/3}}$ ← $f_x(t,x)$ is not continuous at $(0,0)$

Since $f_x(t,x)$ is not continuous at the initial condition, a unique sol'n is not guaranteed by the EUT.

(b) To solve, use SOV $\dfrac{dx}{x + x^{1/3}} = dt$ ← assuming $x \neq 0$

let $u = x^{1/3}$

$\Rightarrow x = u^3$

$dx = 3u^2\,du$

so $\displaystyle\int \frac{dx}{x + x^{1/3}} = \int \frac{3u^2\,du}{u^3 + u} = \int \frac{3u}{u^2+1}\,du$

let $w = u^2 + 1$

$dw = 2u\,du$

$$= \frac{3}{2}\int \frac{dw}{w}$$

$$= \frac{3}{2}\ln|w|$$

w is always positive

$$= \frac{3}{2}\ln(u^2+1)$$

$$= \frac{3}{2}\ln(x^{2/3}+1)$$

So $\dfrac{3}{2}\ln(x^{2/3}+1) = t + c$

$\ln(x^{2/3}+1) = \dfrac{2}{3}t + c$

$x^{2/3} + 1 = ce^{2t/3}$

$x(t) = (ce^{2t/3} - 1)^{3/2}$ ← Notice $x=0$ cannot be achieved for any value of c

$x(0) = (c-1)^{3/2} \overset{IC}{=} 0$ so $c = 1$

specific sol'ns are $x(t) = (e^{2t/3} - 1)^{3/2}$

and

$x(t) = 0$ since

4)

$$\left((t^2-2t+2)D^3 - t^2D^2 + 2tD - 2\right)x = 0 \qquad -\infty < t < \infty$$

(a)

(i) 3

(ii) Yes

(iii) No

(iv) Yes (b/c all coef fxns and source fxn are continuous for all $t \in \mathbb{R}$

and t^2-2t+2 is never zero (the discriminant (b^2-4ac) is negative))

(b) $h_1(t) = t$

$h_1' = 1$

$h_1'' = 0$

$h_1''' = 0$

So let $L = (t^2-2t+2)D^3 - t^2D^2 + 2tD - 2$

then $Lh_1 = 0 - 0 + 2t - 2(t) = 0$

Since $Lh_1 = 0$ h_1 is a sol'n.

$h_2(t) = t^2$

$h_2' = 2t$

$h_2'' = 2$

$h_2''' = 0$

$Lh_2 = 0 - t^2(2) + 2t(2t) - 2(t^2) = -2t^2 + 4t^2 - 2t^2$
$= 0$

So h_2 is a sol'n

$h_3 = e^t$

$h_3' = e^t$

$h_3'' = e^t$

$h_3''' = e^t$

$Lh_3 = (t^2-2t+2)e^t - t^2e^t + 2te^t - 2e^t$
$= 0$

So h_3 is a sol'n

(c) $W[h_1\ h_2\ h_3](t) = \det \begin{bmatrix} t & t^2 & e^t \\ 1 & 2t & e^t \\ 0 & 2 & e^t \end{bmatrix}$

$\Rightarrow W[h_1, h_2, h_3](0) = \det \begin{bmatrix} \cancel{0} & \cancel{0} & \cancel{1} \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} = (-1)\det\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = -2$

Since $W[h_1, h_2, h_3](t_0) \neq 0$ for a t_0 in $-\infty < t < \infty$

Wronskian test asserts $h_1\ h_2\ h_3$ are linearly independent.

(d) $c_1h_1 + c_2h_2 + c_3h_3 = 0$

$\iff c_1t + c_2t^2 + c_3e^t = 0$ (✱)

choose $t=-1 \rightsquigarrow -c_1 + c_2 + c_3e = 0$ (1)

$t=0 \rightsquigarrow c_3 = 0$ (2)

$t=1 \rightsquigarrow c_1 + c_2 + c_3e = 0$ (3)

Eq'n (2) asserts $\boxed{c_3=0}$ Substituting $c_3=0$ in eq'ns (1) and (3) yields

$-c_1 + c_2 = 0$ (1)

$c_1 + c_2 = 0$ (3)

Adding yields $2c_2 = 0$ So $\boxed{c_2=0}$

Substituting $c_2 = c_3 = 0$ in eq'n (1) yields $\boxed{c_1=0}$.

So $c_1=c_2=c_3=0$ is only sol'n to eq'ns (1), (2), and (3); necessary for (✱) to be true.

So h_1, h_2, h_3 are linearly indep.

(e) Since h_1, h_2, h_3 are all sol'ns to $Lx=0$

and h_1, h_2, h_3 are linearly independent

and the order of $Lx=0$ is 3 (same # of linearly indep sol'ns)

the linear combination $x(t) = c_1h_1(t) + c_2h_2(t) + c_3h_3(t)$

$= c_1t + c_2t^2 + c_3e^t$ generates the general sol'n to $Lx=0$.