Mathematics 51 | Tufts University, Department of Mathematics, Midterm Exam 1 | February 19, 2015

Instructions: No calculators, notes or books are allowed. Show all work to receive full credit. Remember to sign your blue book, indicating that you have neither given nor received assistance on this exam.

1. (20 points) Give the general solution to the ordinary differential equation,
$$(D-1)^2(D+2)(D^2-4)^2x = 0.$$

2. (30 points) The ordinary differential equation
$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{x}{t}+1\right)$$
arises in the theory of shock waves. The function $x(t)$ gives the position of the shock wave as a function of time. You can assume $t > 0$.
   (a) (20 points) Find the general solution of this differential equation.
   (b) (10 points) Suppose we know that the position of the shock wave at time $t = 1$ is $x(1) = 0$. Find the specific solution giving the position of the shock wave as a function of time thereafter.

3. (20 points) Consider the ordinary differential equation
$$\frac{dx}{dt} = x + x^{1/3},$$
with initial condition $x(0) = 0$.
   (a) (10 points) Does the Existence and Uniqueness Theorem guarantee the existence of a unique solution to this initial-value problem? Why or why not?
   (b) (10 points) If you think the solution is unique, find it. Otherwise, find two solutions. (Hint: The substitution $u = x^{1/3}$ will help with the integral over x that you encounter.)

4. (30 points) This problem concerns the differential equation
$$\left[(t^2-2t+2)D^3 - t^2D^2 + 2tD - 2\right]x = 0.$$
for $-\infty < t < +\infty$.
   (a) (8 points, no partial credit)
      i. What is the order of this equation?
      ii. Is this equation linear?
      iii. Does it have constant coefficients?
      iv. Is it normal for all real t?
   (b) (6 points) Show that the functions
$$\begin{aligned} h_1(t) &= t \\ h_2(t) &= t^2 \\ h_3(t) &= e^t \end{aligned}$$
   solve the given equation.
   (c) (6 points) Use the Wronskian test to determine whether or not the given solutions are linearly independent.
   (d) (6 points) Use the definition of linear independence to determine whether or not the given solutions are linearly independent.
   (e) (4 points) Does a linear combination of the given solutions constitute the general solution? Why or why not?