Math 51
Differential Equations

Tufts University
Department of Mathematics
Exam 2

Monday, April 7, 2014
12:00–1:20pm

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. A correct answer with no work may not necessarily score any points. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature, you are pledging that you have neither given nor received assistance on the exam. Any violations will be reported to the appropriate dean, and will result in an F for the course.

1. (20 points) Consider the equation $(D^2 - 4)x = \begin{cases} 2 & t < 2 \\ 0 & t \geq 2 \end{cases} \qquad x(0) = 1, x'(0) = 0$
   (a) Express the right-hand side in terms of functions $u_a(t)$.
   (b) Find the Laplace transform of the right-hand side.
   (c) Find the Laplace transform of the left-hand side and solve for $\mathscr{L}x$
   (d) Find $x(t)$

2. (10 points) Find $\mathscr{L}^{-1}\left(\dfrac{s}{s^2 + 4s + 5}\right)$

3. (10 points) Consider the vectors

$$\vec{u_1} = \begin{pmatrix} 1 \\ 5 \\ -1 \\ 0 \\ 2 \end{pmatrix} \qquad \vec{u_2} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{u_3} = \begin{pmatrix} -1 \\ 9 \\ -11 \\ -4 \\ 4 \end{pmatrix}$$

   (a) Reduce the matrix with columns $\vec{u_1}, \vec{u_2}$, and $\vec{u_3}$.
   (b) Conclude whether $\vec{u_1}, \vec{u_2}$, and $\vec{u_3}$ are independent or not. You must give reasons.

4. (15 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

   (a) Find the eigenvalues of A and their corresponding eigenvectors.
   (b) Find the general solution of $D\vec{x} = A\vec{x}$

5. (15 points) Consider $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$
   (a) Find the characteristic polynomial and the eigenvalues of A.
   (b) Find the general solution of $D\vec{x} = A\vec{x}$.

6. (10 points) $\lambda = -1$ is a triple eigenvalue associated to

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

You don't have to verify this. Find the generalized eigenvectors associated to this eigenvalue and the general solution of $D\vec{x} = A\vec{x}$.

7. (20 points) Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ and $\vec{E} = \begin{pmatrix} e^{-t} \\ 1 \end{pmatrix}$

Solve the non-homogeneous system $D\vec{x} = A\vec{x} + \vec{E}$.