Name:____

Professors name:____

Due date: 9/18

1. **The Hailstone Sequence:**

   In this question we look at the hailstone sequence. First we construct an intermediate sequence, $\{a_n\}_{n=1}^{\infty}$. Let $a_1 = k$ for some positive integer k. The initial term k is the "seed" of the intermediate sequence (not the hailstone sequence yet). Generate the rest of the sequence as follows:

   $$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } \quad a_n \text{ is even} \\ 3a_n + 1 & \text{if } \quad a_n \text{ is odd} \end{cases}$$

   We stop counting terms in the sequence, if any term equals 1. An example for $k = 3$ is

   $$a_1 = 3,\ a_2 = 10,\ a_3 = 5,\ a_4 = 16,\ a_5 = 8,\ a_6 = 4,\ a_7 = 2,\ a_8 = 1.$$

   The hailstone sequence $\{H_k\}_{k=1}$ is defined by letting H_k be the number of terms in the sequence a_n up to and including the term equal to 1, with seed k. For example, by counting the terms in the example above, $H_3 = 8$.

   (a) Give the 10 sequences, a_n, obtained by letting $k = 1, 2, 3, 4, ..., 8, 9, 10$.

   (b) Give the first 10 terms of the hailstone sequence.

   (c) Plot on a graph as many terms of the hailstone sequence as you can. You can use excel, or some other program, to generate terms if you like, you may attach a print out rather than writing them. Why do you think the sequence is called the hailstone sequence?

   Fun Fact: It is an open problem (the answer is unknown) whether H_k is defined for each seed (does the sequence terminate when started with seed a_0 for all possible a_0?) The Ulam or Collatz conjecture is the hypothesis that H_k exists for each a_0.

2. Consider the following sequences generated by a given differentiable function, $f(x)$:

   $$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}.$$

   (a) Write down the recurrence relation of the above sequence for the following functions:

   i. $f(x) = x^4 + x - 1$ ii. $f(x) = \sin(\pi x)$

   (b) Next, consider the function $f(x) = \frac{1}{x} - 2$. Determine in whatever way possible if the sequence converges, and if so to what, for the following starting terms:

   i. $a_1 = \dfrac{3}{4}$ ii. $a_1 = 2$ iii. $a_1 = 0$

   As you can see it matters what you start with. We'll discuss this sequence in more detail later in the semester.

3. Determine if the following sequences converge and, if so, to what.

   (a) $a_n = \left(1 - \frac{3}{n}\right)^{n^2}$

   (b) $b_n = \frac{(-1)^n}{6^{n+3}}$