

(1)

3. For $n \geq 9$, $0 \leq \frac{8^n}{n!} = \frac{8}{n} \cdot \frac{8}{n-1} \cdots \frac{8}{9} \frac{8}{8} \frac{8}{7} \cdots \frac{8}{2} \frac{8}{1} \leq \frac{8}{n} \cdot \frac{8^8}{8!} \rightarrow 0$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{8^n}{n!} = 0 \text{ by the square theorem}$$

5 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} n^{\frac{1}{n}}$

Consider $\ln n^{\frac{1}{n}} = \frac{1}{n} \ln n$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} \stackrel{\text{H}\ddot{\text{o}}\text{o}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = e^0 = 1$$

7. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{\frac{1}{\ln n}}$

Consider $\ln \left(\frac{1}{n}\right)^{\frac{1}{\ln n}} = \frac{1}{\ln n} \ln \left(\frac{1}{n}\right)$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n} \ln \left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{-\ln n}{\ln n} = -1$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{\frac{1}{\ln n}} = e^{-1}$$

9. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n}{(0.9)^n} = \lim_{n \rightarrow \infty} \left(-\frac{10}{9}\right)^n$

Since $\left|-\frac{10}{9}\right| > 1$, the sequence diverges.

(2)

53. $a_k = \frac{1}{k}$

$\lim_{k \rightarrow \infty} a_k = 0$ but $\sum_{k=1}^{\infty} a_k$ diverges (Harmonic series)

54. This is impossible.

$\sum_{k=1}^{\infty} a_k$ converges $\iff \lim_{k \rightarrow \infty} a_k = 0$

55(a) $\lim_{k \rightarrow \infty} \frac{k}{k+1} = \lim_{k \rightarrow \infty} \frac{1}{1 + \frac{1}{k}} = 1$

(b) because $\lim_{k \rightarrow \infty} a_k = 1$. series $\sum_{k=1}^{\infty} a_k$ diverges by the divergence test

56 No. when $r = 1$

$\lim_{k \rightarrow \infty} r^k = 1$ geometric sequence converges

However,

$\sum_{k=1}^{\infty} r^k$ diverges because $r = 1$

(3)

$$13. \sum_{k=1}^{\infty} 3(1.001)^k = 3 \sum_{k=1}^{\infty} (1.001)^k$$

because $|r| = 1.001 > 1$. the geometric series diverges.

$$17. \sum_{k=1}^{\infty} \left(\frac{3}{3k-2} - \frac{3}{3k+1} \right)$$

$$S_n = (3 - \cancel{\frac{3}{4}}) + (\cancel{\frac{3}{4}} - \cancel{\frac{3}{7}}) + (\cancel{\frac{3}{7}} - \cancel{\frac{3}{10}}) + \dots$$

$$+ (\cancel{\frac{3}{3n-5}} - \cancel{\frac{3}{3n-2}}) + (\cancel{\frac{3}{3n-2}} - \cancel{\frac{3}{3n+1}})$$

$$S_n = 3 - \frac{3}{3n+1}$$

$$\sum_{k=1}^{\infty} \left(\frac{3}{3k-2} - \frac{3}{3k+1} \right) = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(3 - \frac{3}{3n+1} \right) = 3$$

$$19. \sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}} = \sum_{k=1}^{\infty} \frac{1}{3^2} \frac{2^k}{3^k} = \frac{1}{9} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$$

$$= \frac{1}{9} \cdot \frac{2}{3} \cdot \frac{1}{1-\frac{2}{3}} = \frac{2}{9}$$

(4)

$$23 \quad \sum_{k=1}^{\infty} k^{-\frac{2}{3}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}}$$

it is a p-series with $p = \frac{2}{3} \neq 1$
 \Rightarrow it diverges

$$25 \quad \sum_{k=1}^{\infty} \frac{2^k}{e^k} = \sum_{k=1}^{\infty} \left(\frac{2}{e}\right)^k$$

it is a geometric series with $|r| = \frac{2}{e} < 1$
 \Rightarrow it converges

$$27. \quad \sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$$

$$L = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{2^{k+1} (k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{2^k k!}$$

$$= \lim_{k \rightarrow \infty} 2 \left(\frac{k}{k+1}\right)^k = 2 \lim_{k \rightarrow \infty} \left(1 - \frac{1}{k+1}\right)^k$$

$$\text{consider } \ln\left(1 - \frac{1}{k+1}\right)^k = k \ln\left(1 - \frac{1}{k+1}\right)$$

$$\lim_{k \rightarrow \infty} k \ln\left(1 - \frac{1}{k+1}\right) = \lim_{x \rightarrow \infty} \frac{\ln\left(1 - \frac{1}{x+1}\right)}{\frac{1}{x}}$$

$$\stackrel{0}{\underline{0}} \lim_{x \rightarrow \infty} \frac{\frac{1}{1-x+1} \cdot \left(\frac{1}{x+1}\right)^{-2}}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{1}{1-\frac{1}{x+1}} \cdot (-1) \cdot \left(1 + \frac{1}{x}\right)^{-2}$$

$$= -1$$

$$\Rightarrow L = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = 2 \cdot e^{-1} = \frac{2}{e} < 1$$

\Rightarrow the series converges by the ratio test

(5)

29

$$\sum_{k=1}^{\infty} \frac{3}{2+e^k}$$

$$a_k = \frac{3}{2+e^k}, \text{ consider } b_k = \frac{3}{e^k}$$

$\sum_{k=1}^{\infty} b_k$ converges because it is a geometric series

with $|r| = \frac{1}{e} < 1$

$$0 < a_k = \frac{3}{2+e^k} < b_k = \frac{3}{e^k}$$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ converges by the ordinary comparison test

30

$$\sum_{k=1}^{\infty} k \sin \frac{1}{k}$$

$$\lim_{k \rightarrow \infty} k \sin \frac{1}{k} = \lim_{x \rightarrow 0} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \stackrel{\text{H}\ddot{\text{o}}\text{pital}}{=} \lim_{x \rightarrow 0} \frac{\cos(\frac{1}{x})(-\frac{1}{x^2})}{(-\frac{1}{x^2})} = \lim_{x \rightarrow 0} \cos(\frac{1}{x}) = 1$$

\Rightarrow the series diverges by the divergence test.

31

$$\sum_{k=1}^{\infty} \frac{\sqrt[4]{k}}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^{3-\frac{1}{4}}}$$

consider $b_k = \frac{1}{k^2}$, $\sum_{k=1}^{\infty} b_k$ is a convergent p-series with $p=2>1$

For $k \geq 2$

$$0 < a_k = \frac{1}{k^{3-\frac{1}{4}}} < b_k = \frac{1}{k^2}$$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ converges by the ordinary comparison test.

(6)

33

$$\sum_{k=1}^{\infty} k^5 e^{-k}$$

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{(k+1)^5 e^{-(k+1)}}{k^5 e^{-k}} = \lim_{k \rightarrow \infty} \frac{1}{e} \left(\frac{k+1}{k}\right)^5 \\ &= \lim_{k \rightarrow \infty} \frac{1}{e} \left(\frac{1 + \frac{1}{k}}{1}\right)^5 = \frac{1}{e} < 1 \end{aligned}$$

\Rightarrow the series converges by the ratio test

35

$$\sum_{k=1}^{\infty} \frac{\ln k^2}{k^2}$$

choose $b_k = \frac{2}{k^{\frac{3}{2}}}$ $\sum_{k=1}^{\infty} b_k$ is a convergent p-series with $p = \frac{3}{2} > 1$

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\frac{2 \ln k / k^2}{2/k^{\frac{3}{2}}}}{2/k^{\frac{1}{2}}} = \lim_{k \rightarrow \infty} \frac{\ln k}{k^{\frac{1}{2}}} \\ &= \lim_{x \rightarrow \infty} \frac{\ln x}{x^{\frac{1}{2}}} \stackrel{\text{H}\ddot{\text{o}}\text{p}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x \rightarrow \infty} 2x^{-\frac{1}{2}} = 0 \end{aligned}$$

$0 = L < \infty \Rightarrow$ the series converges by the limit comparison test.

37.

$$\sum_{k=0}^{\infty} \frac{2 \cdot 4^k}{(2k+1)!}$$

$$L = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{\frac{2 \cdot 4^{k+1}}{(2k+3)!}}{\frac{2 \cdot 4^k}{(2k+1)!}} = \lim_{k \rightarrow \infty} \frac{4}{(2k+3)(2k+2)} = 0$$

$L = 0 < \cancel{<} 1 \Rightarrow$ the series converges by the ratio test

(7)

$$43. \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$$

$$\sum_{k=2}^{\infty} |a_k| = \sum_{k=2}^{\infty} \frac{1}{k^2-1} = \frac{1}{2} \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1} \right)$$

$$S_n = \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k+1} \right) \\ = (1 - \cancel{\frac{1}{3}}) + \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} \right) + \dots$$

$$+ \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n-1}} \right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n} \right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1} \right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \right) = \frac{3}{2}$$

$$\Rightarrow \sum_{k=2}^{\infty} |a_k| = \frac{1}{2} \lim_{n \rightarrow \infty} S_n = \frac{3}{4} \text{ converges}$$

$\Rightarrow \sum_{k=2}^{\infty} a_k$ converges absolutely.

$$45. \sum_{k=1}^{\infty} (-1)^k k e^{-k}$$

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} k e^{-k}$$

$$L = \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \rightarrow \infty} \frac{(k+1) e^{-(k+1)}}{k e^{-k}} = \frac{1}{e} \lim_{k \rightarrow \infty} \frac{k+1}{k}$$

$$L = \frac{1}{e} \lim_{k \rightarrow \infty} \frac{1 + \frac{1}{k}}{1} = \frac{1}{e} < 1$$

$\Rightarrow \sum_{k=1}^{\infty} |a_k|$ converges by the ratio test

$\Rightarrow \sum_{k=1}^{\infty} a_k$ converges absolutely.

(8)

$$47 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 10^k}{k!}$$

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{10^k}{k!}$$

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \rightarrow \infty} \frac{\frac{10^{k+1}}{(k+1)!}}{\frac{10^k}{k!}} = \lim_{k \rightarrow \infty} \frac{10}{k+1} = 0 < 1$$

$\Rightarrow \sum_{k=1}^{\infty} |a_k|$ converges by the ratio test

$\Rightarrow \sum_{k=1}^{\infty} a_k$ converges absolutely.

$$48 \sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$$

$$\sum_{k=2}^{\infty} |a_k| = \sum_{k=2}^{\infty} \frac{1}{k \ln k} \quad \text{choose } f(x) = \frac{1}{x \ln x} > 0 \text{ and is continuous}$$

$$f'(x) = \frac{-(\ln x + 1)}{x^2 (\ln x)^2} < 0 \text{ for } x \geq 2 \Rightarrow f(x) \text{ decreases for } x \geq 2$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{C \rightarrow \infty} \int_2^C \frac{1}{x \ln x} dx$$

$$= \lim_{C \rightarrow \infty} \int_{\ln 2}^{\ln C} u^{-1} du = \lim_{C \rightarrow \infty} [u] \Big|_{\ln 2}^{\ln C}$$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$x=2 \Rightarrow u=\ln 2$$

$$x=C \Rightarrow u=\ln C$$

$$= \lim_{C \rightarrow \infty} (\ln(\ln C) - \ln(\ln 2)) = \infty$$

$\Rightarrow \sum_{k=2}^{\infty} |a_k|$ diverges by the integral test

On the other hand, consider $\sum_{k=2}^{\infty} a_k$.

because $f(x)$ decreases for $x \geq 2 \Rightarrow 0 < \frac{1}{(k+1)\ln(k+1)}$

$$\lim_{k \rightarrow \infty} \frac{1}{k \ln k} = 0 \quad \left. \begin{aligned} 0 < \frac{1}{(k+1)\ln(k+1)} &< \frac{1}{k \ln k} \end{aligned} \right\} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges by} \\ \text{the alternating series test}$$

Since the series is not absolutely convergent, but still converges, then overall. $\sum_{k=1}^{\infty} a_k$ converges conditionally.

(9)

$$49 \quad \sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^{k+1}}{k^2}$$

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{2^{k+1}}{k^2}$$

$$L = \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \rightarrow \infty} \frac{2^{k+2}/(k+1)^2}{2^{k+1}/k^2} = \lim_{k \rightarrow \infty} 2 \cdot \left(\frac{k}{k+1}\right)^2$$

$$L = 2 \cdot \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^2 = 2 > 1$$

$\Rightarrow \sum_{k=1}^{\infty} |a_k|$ diverges by the ratio test

Because the ratio test is used to show $\sum_{k=1}^{\infty} |a_k|$ diverges, $\Rightarrow \sum_{k=1}^{\infty} a_k$ diverges.