Math 34: Guidelines for Applying Convergence Tests

SEQUENCES

a) You may use $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad |\text{r}| < 1 \\ 1 & \text{if} \quad \text{r} = 1 \\ \text{diverges} & \text{if} \quad |\text{r}| > 1 \quad (\text{ tends to } +\infty \text{ if r} > 1) \end{cases}$

b) If you wish to use L'Hôpital's Rule to find a limit of a sequence, switch to the continuous variable x. Then indicate that you are using L'Hôpital's Rule and that you know the hypothesis under which it is being used in the given problem. The following example shows one way to make these indications:

$$\lim_{n\to\infty} \frac{\ln n}{n} \underset{\text{switch to } x}{=} \lim_{x\to\infty} \frac{\ln x}{x} \underset{\frac{\text{l'Hop.}}{\infty/\infty}}{=} \lim_{x\to\infty} \frac{1}{x} = 0$$

c) Don't replace algebra or calculus with *ad hoc* rules for computing limits or explanations involving growth rates. For example, write

$$\lim_{x\to\infty} \frac{2x^3 - x}{3x^3 + 2} = \lim_{x\to\infty} \frac{2 - 1/x^2}{3 + 2/x^3} = \frac{2}{3}$$

even though you may know a rule which says that you should divide leading coefficients of polynomials for such limits.

SERIES

a) Standard Math 12 shorthand for sequences and series:

- C's - converges
- D's - diverges
- OCT- (Ordinary) Comparison Test
- LCT - Limit Comparison Test
- AST - Alternating Series Test
- RATFACE - Ratio Test for Absolute Convergence
- ROOTFACE - Root Test for Absolute Convergence
- Div Test - Divergence Test
- AbC - absolute convergence
- CC - conditional convergence

b) Complete justification must be given to establish convergence or divergence of a series. In particular, you must name the test you are using (it is OK to use one of the acronyms), show that its hypotheses are satisfied, and draw a correct conclusion.

c) If you want to show that the series $\sum_{n=n_0}^{\infty} a_n$ converges by the Integral Test, you must set $f(n) = a_n$, extend this to a function $f(x)$ of a continuous variable for all $x \geq n_0$, and:

- confirm that $f(x)$ is continuous and $f(x) > 0$ for all $x \geq n_0$;
- show that $f(x)$ is eventually decreasing

You may then apply the Integral Test. Of course, you must correctly investigate $\int_{n_0}^{\infty} f(x)dx$.

d) Comparison Tests.

- In order to use any of the comparison tests, the terms of both series must be positive.

- You must explain why the series to which you are comparing your series converges or diverges.

  Here are a few examples of such explanations:

  - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series because $|r| = |\frac{1}{2}| < 1$, or

  - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series with $p = 3/2 > 1$.

- If you use the Ordinary Comparison Test (OCT), you must establish the appropriate inequality between the terms of the two series.

- If you wish to use the Limit Comparison Test (LCT) for the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, you must show that the limit $L = \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and positive, and you must say: because $0 < L < \infty$, the LCT may be applied.

f) Ratio Test. To apply the ratio test for absolute convergence (known as RATFACE, or simply the ratio test) to the series $\sum_{n=1}^{\infty} a_n$, you must:

- Find $r = \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$,

- compare this limit r to 1, and draw the appropriate conclusion about whether it converges absolutely or diverges. Recall that the series converges absolutely when $r < 1$, the series diverges when $r > 1$, and the ratio test does not permit you to draw any conclusion at all when r=1. (If the original series is a positive series then convergence and absolute convergence are equivalent.)

e) Root Test. To apply the root test for absolute convergence (known as ROOTFACE, or simply the root test) to the series $\sum_{n=1}^{\infty} a_n$, you must:

- Find $\rho = \displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|}$,

- compare this limit ρ to 1, and draw the appropriate conclusion about whether it converges absolutely or diverges. Recall that the series converges absolutely when $\rho < 1$, the series diverges when $\rho > 1$, and the root test does not permit you to draw any conclusion at all when $\rho = 1$ (If the original series is a positive series then convergence and absolute convergence are equivalent.)

g) In order to correctly apply the Alternating Series Test (AST) for the series of the form

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{\mathrm{n}=1}^{\infty}(-1)^{\mathrm{n}+1}\mathrm{b_n}$$

with b_n positive, you must do the following:

- Show that $\lim_{n\to\infty} b_n = 0$.
- Show that the sequence $\{b_n\}$ is eventually nonincreasing. Sometimes it is possible to verify that $b_{n+1} \leq b_n$ for all n sufficiently large by simple algebra. However, you must show this is true in general and not just for the first few terms of the sequence. If it is not possible to do so, you must let $f(n) = b_n$, extend this to a function $f(x)$ of a continuous variable, and show that $f'(x) \leq 0$ eventually, i.e. for all $x \geq x_0$.
- If $\lim_{n\to\infty} b_n \neq 0$, you must state $\lim_{n\to\infty}(-1)^n b_n \neq 0$ or $\lim_{n\to\infty}(-1)^{n+1} b_n \neq 0$ (oscillates) and then conclude the series diverges by the Divergence Test.

h) You should know and understand the Remainder Estimate for an Alternating Series:

$$|\mathrm{Error}| = |\mathrm{S} - \mathrm{S_n}| \leq \mathrm{b_{n+1}}$$

where S is the actual sum of the alternating series and $\{S_n\}$ is the sequence of its partial sums.

i) Know that a series $\sum_{n=1}^{\infty} a_n$ is

- *Absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.
- *Conditionally convergent* if it is convergent but not absolutely convergent.
- In particular, if $\{b_n\}$ is a sequence of *positive* numbers, the alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ is absolutely convergent provided that $\sum_{n=1}^{\infty} b_n$ is convergent.
- To show that an infinite series is conditionally convergent, **you must do two things:** first, show that it is *not* absolutely convergent, and then show that it is convergent.
- If you used either RATFACE or ROOTFACE to determine that $\sum_{n=1}^{\infty} a_n$ is not absolutley convergent, the series *diverges* and there is no chance that it can be conditionally convergent.