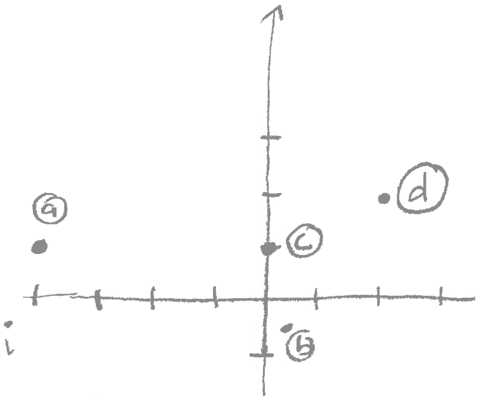
MATH 34 Complex Numbers



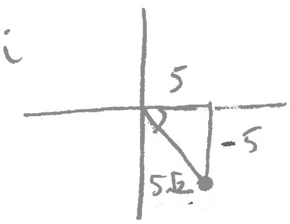
① a) $(3+2i) - (i+7) = -4+i$

b) $\frac{1}{1+i} = \frac{1(1-i)}{(1+i)(1-i)} = \frac{1-i}{1-i^2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$

c) i^{441} 441 ÷ 4 has a remainder of 1 so $i^{441} = i^1 = i$

d) $\overline{2i(i^2-i)} = \overline{2i^3 - 2i^2} = \overline{-2i+2} = \overline{2-2i} = 2+2i$

② a) $z = 5-5i$

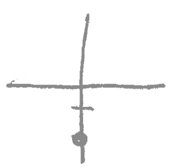


$r = \sqrt{25+25} = \sqrt{25 \cdot 2} = 5\sqrt{2}$

$\left.\begin{array}{l} \sin\theta = -5/5\sqrt{2} = -1/\sqrt{2} = -\sqrt{2}/2 \\ \cos\theta = 5/5\sqrt{2} = 1/\sqrt{2} = \sqrt{2}/2 \end{array}\right\} \theta = \frac{-\pi}{4}$ (or $7\pi/4$)

$$z = 5\sqrt{2}\, e^{i(-\pi/4)}$$

b) $z = -2i$

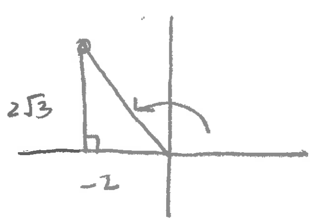


$r = 2$

$\theta = 3\pi/2$ (or $-\pi/2$)

$z = 2e^{i(-\pi/2)}$

c) $z = -2 + 2\sqrt{3}i$



$r = \sqrt{4+12} = \sqrt{16} = 4$

$\left.\begin{array}{l} \sin\theta = 2\sqrt{3}/4 = \sqrt{3}/2 \\ \cos\theta = -2/4 = -1/2 \end{array}\right\} \theta = 2\pi/3$

$$z = 4e^{i(2\pi/3)}$$

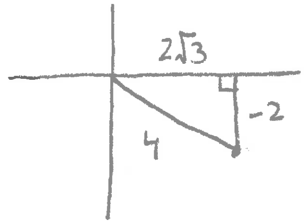
③ a) $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + 0 = 1$

b) $10e^{\pi + i\pi/4} = 10e^{\pi}e^{i\pi/4} = 10e^{\pi}[\cos \pi/4 + i\sin \pi/4] = 10e^{\pi}[\sqrt{2}/2 + i\sqrt{2}/2]$
$= 5e^{\pi}\sqrt{2} + i5e^{\pi}\sqrt{2}$

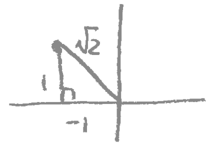
c) $4e^{i\pi/3} = 4[\cos \pi/3 + i\sin \pi/3] = 4[\frac{1}{2} + i\sqrt{3}/2] = 2 + i2\sqrt{3}$

④ $z = 2\sqrt{3} - 2i$ and $w = -1 + i$

z:



w:



$r = \sqrt{12+4} = \sqrt{16} = 4$

$\left.\begin{array}{l} \sin\theta = -1/2 \\ \cos\theta = 2\sqrt{3}/4 = \sqrt{3}/2 \end{array}\right\} \theta = -\pi/6$

$z = 4e^{i(-\pi/6)}$

$r = \sqrt{2}$

$\left.\begin{array}{l} \sin\theta = 1/\sqrt{2} = \sqrt{2}/2 \\ \cos\theta = \frac{-1}{\sqrt{2}} = -\sqrt{2}/2 \end{array}\right\} \theta = 3\pi/4$

$w = \sqrt{2}\, e^{i(3\pi/4)}$

$$zw = \left[4e^{i(-\pi/6)}\right]\left[\sqrt{2}\, e^{i(3\pi/4)}\right] = 4\sqrt{2}\, e^{i(-\pi/6 + 3\pi/4)} = 4\sqrt{2}\, e^{i(7\pi/12)}$$

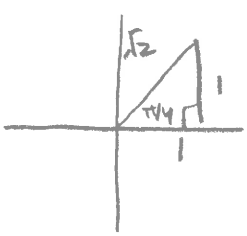
$$\frac{1}{z} = (z)^{-1} = \left(4e^{i(-\pi/6)}\right)^{-1} = 4^{-1} e^{i(-\pi/6)(-1)} = \frac{1}{4} e^{i\pi/6}$$

⑤ (a) $(2e^{i\pi/3})^5 = 2^5 e^{(i\pi/3)(5)} = 32 e^{i(5\pi/3)} = 32\left[\cos 5\pi/3 + i \sin 5\pi/3\right]$

$= 32\left[\left(\frac{1}{2}\right) + i\left(-\sqrt{3}/2\right)\right]$

$= 16 - i\, 16\sqrt{3}$

(b) $(1+i)^{20} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{20} = (\sqrt{2})^{20}\, e^{i\,5\pi}$



$= 2^{10} e^{i\pi}$

$= 1024\left[\cos\pi + i \sin\pi\right]$

$= 1024\left[-1\right] = -1024$

⑥

$(e^{i\theta})^3 = e^{i3\theta} = \underline{\underline{\cos 3\theta}} + \underline{i \sin 3\theta}$

$(e^{i\theta})^3 = [\cos\theta + i\sin\theta]^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3(\cos\theta)(i^2\sin^2\theta) + i^3\sin^3\theta$

$= \cos^3\theta + i(3\cos^2\theta\sin\theta) - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \underline{\underline{\cos^3\theta - 3\cos\theta\sin^2\theta}} + i\underline{[3\cos^2\theta\sin\theta - \sin^3\theta]}$

Equate real parts: $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Equate imaginary parts: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

⑦ a) Find eight roots of $w = 1$

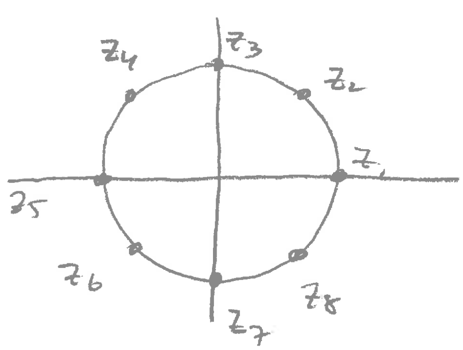
$r = 1, \quad \theta = 0$

Eight roots on a circle of radius $\sqrt[8]{1} = 1$, spaced $2\pi/8 = \pi/4$ apart

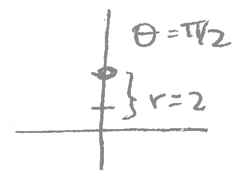
First ∡ is $0/8 = 0$

Roots are:

$\underset{z_1}{1}, \underset{z_2}{e^{i\pi/4}}, \underset{z_3}{e^{i\pi/2}}, \underset{z_4}{e^{i3\pi/4}}, \underset{z_5}{e^{i\pi}}, \underset{z_6}{e^{i5\pi/4}}, \underset{z_7}{e^{i3\pi/2}}, \underset{z_8}{e^{i7\pi/4}}$



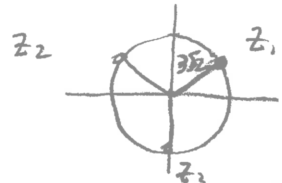
b) Cube roots of $w = 2i$



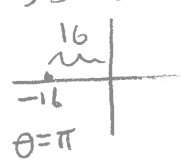
3 roots spaced $2\pi/3$ apart on a circle of radius $\sqrt[3]{2}$

First ∡ is $(\pi/2)/3 = \pi/6$

$z_1 = \sqrt[3]{2}\, e^{i\pi/6}$, $z_2 = \sqrt[3]{2}\, e^{i5\pi/6}$, $z_3 = \sqrt[3]{2}\, e^{i(3\pi/2)}$



c) Solns to $z^4 = -16$: all 4^{th} roots of $w = -16$



4 roots spaced $\frac{2\pi}{4} = \frac{\pi}{2}$ apart on a circle of radius $\sqrt[4]{16} = 2$

First ∡ is $\pi/4$

$z_1 = 2e^{i\pi/4}$, $z_2 = 2e^{i3\pi/4}$

$z_3 = 2e^{i5\pi/4}$, $z_4 = e^{i7\pi/4}$