Math 34
Calculus II

TUFTS UNIVERSITY
Department of Mathematics
Final Exam

May 2, 2014
All sections

No books, notes, **or calculators**. **TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You are not allowed to quote results about growth rates. You are required to **sign** your exam book. With your signature, you pledge that you have neither given nor received assistance on this exam.

Problem	Point Value	Points
1	12	
2	8	
3	12	
4	7	
5	8	
6	8	
7	6	
8	8	
9	12	
10	5	
11	6	
12	8	
	100	

KEY

1. (12 points) Integrate.

(a) $\int \cos^3(x)\, dx$ $= \int \cos^2 x \cos x\, dx = \int (1-\sin^2 x) \cos x\, dx$

$\left(\begin{array}{l} u = \sin x \\ du = \cos x\, dx \end{array} \right)$ $= \int (1-u^2)\, du$

$= u - \dfrac{u^3}{3} + C$

$= \boxed{\sin x + \dfrac{\sin^3 x}{3} + C}$

(b) $\int \dfrac{dx}{x^2 - 3x - 4}$ $= \int \dfrac{dx}{(x-4)(x+1)}$

$\dfrac{1}{(x-4)(x+1)} = \dfrac{A}{x-4} + \dfrac{B}{x+1}$

$1 = A(x+1) + B(x-4)$

$x = -1 \quad 1 = B(-5) \Rightarrow B = -1/5$

$x = 4 \quad 1 = A(5) \Rightarrow A = 1/5$

$= \frac{1}{5} \int \dfrac{dx}{x-4} - \frac{1}{5} \int \dfrac{dx}{x+1}$

$= \boxed{\frac{1}{5} \ln|x-4| - \frac{1}{5} \ln|x+1| + C}$

2. (8 points) Find the limits of each of the following sequences or state that they do not exist.

(a) $a_n = \dfrac{5\sin n}{n^3}$.

$\lim_{n\to\infty} \dfrac{5\sin n \to \text{oscillate}}{n^3 \to \infty}$

SQUEEZE Theorem

$\dfrac{-5}{n^3} \leq \dfrac{5\sin n}{n^3} \leq \dfrac{5}{n^3}$, $n \geq 1$

$\to 0 \leftarrow$

$\boxed{\text{Sequence converges to } 0.}$

(b) $a_n = n\ln\left(1+\dfrac{4}{n}\right)$.

$\lim_{n\to\infty} \underbrace{n}_{\to\infty} \underbrace{\ln\left(1+\frac{4}{n}\right)}_{\to 0} = \lim_{n\to\infty} \dfrac{\ln\left(1+\frac{4}{n}\right) \to 0}{\frac{1}{n} \to 0} = \lim_{x\to\infty} \dfrac{\ln\left(1+\frac{4}{x}\right) \to 0}{\frac{1}{x} \to 0} \overset{L}{=}$

$\lim_{x\to\infty} \dfrac{\left[\frac{1}{(1+4/x)}\right]\left[\frac{-4}{x^2}\right]}{\left[-\frac{1}{x^2}\right]} = \lim_{x\to\infty} \dfrac{4}{1+4/x} = 4$

$\boxed{\text{Sequence converges to } 4}$

3. (12 points) Determine whether each of the following series converges or diverges. Justify your answer. State and check hypotheses of any test, rules or theorems you use. You may *not* simply quote a theorem.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{24k^2 + 30k}{k^3 + 1} \sim \sum_{k=1}^{\infty} \frac{k^2}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k}$

positive Σ

LCT $\displaystyle L = \lim_{k\to\infty} \frac{24k^2+30k}{k^3+1} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{24k^3 + 30k^2}{k^3+1}$

$\displaystyle = \lim_{k\to\infty} \frac{24 + 30/k}{1 + 1/k^3} = 24.$

Since $0 < 24 < +\infty$ and since $\sum \frac{1}{k}$ is the divergent H-Σ, our Series D's by the LCT.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{10e^k}$

A.S. Test

① $\displaystyle \lim_{k\to\infty} \frac{k}{10e^k} = \lim_{x\to\infty} \frac{x}{10e^x} \overset{L}{=} \lim_{x\to\infty} \frac{1}{10e^x} = 0$ ✓

② $\displaystyle \frac{k}{10e^k} > 0$ for $k \geq 1$

③ Need to Show $\left\{\frac{k}{10e^k}\right\}$ is eventually decreasing:

Let $\displaystyle f(x) = \frac{x}{10e^x}$. $\displaystyle f'(x) = \frac{10e^x - x \cdot 10e^x}{(10e^x)^2} = \frac{10e^x(1-x)}{(10e^x)^2} < 0$ for $x > 1$

$\therefore \left\{\frac{k}{10e^k}\right\}$ is decreasing for $k > 1$.

$\therefore$ This series passes the A.S. Test & hence converges.

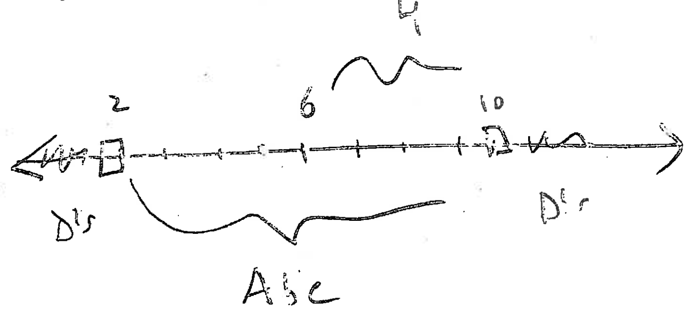
4. (7 points) Find the radius of convergence and interval of convergence for the following power series:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-6)^k}{4^k \cdot k} \qquad \text{Center } x=6$$

$\boxed{1}$ RATIO TEST $\quad r = \lim_{k\to\infty} \left| \frac{(x-6)^{k+1}}{4^{k+1}(k+1)} \cdot \frac{4^k \cdot k}{(x-6)^k} \right|$

$$= \frac{|x-6|}{4} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x-6|}{4}$$

$\boxed{2}$ Solve $\frac{|x-6|}{4} < 1$



$|x-6| < 4$

$-4 < x-6 < 4$

$2 < x < 10$

$$\boxed{\text{ROC} = 4}$$

$\boxed{3}$ Endpts (i) $x=2$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-4)^k}{4^k \cdot k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1} 4^k}{4^k \cdot k} = -\sum_{k=1}^{\infty} \frac{1}{k}$

This is $-(H\text{-}S)$. It diverges.

(ii) $x=10$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 4^k}{4^k \cdot k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4^k}$

This is the ALT H-S. It converges

$$\boxed{\text{IOC} = (2, 10]}$$

5. (8 points) Compute the Taylor series for

$$f(x) = \frac{1}{(x+3)^2}$$

centered at $a = -2$ **using the definition of the Taylor series**. Write the series using summation notation. (You **do not** need to find the radius of convergence or the interval of convergence.)

n	$f^{(n)}(x)$	$f^{(n)}(-2)$
0	$(x+3)^{-2}$	1
1	$-2(x+3)^{-3}$	-2
2	$2\cdot3(x+3)^{-4}$	$2\cdot3$
3	$-2\cdot3\cdot4(x+3)^{-5}$	$-2\cdot3\cdot4$
$\vdots$		$\vdots$
n		$(-1)^n(n+1)!$

$$\text{TS}: \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!\,(x+2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x+2)^n$$

6. (8 points) Consider the following parametric equations:
$x = e^t$ and $y = 4 - e^{2t}$; $-\infty < t \leq \ln 2$

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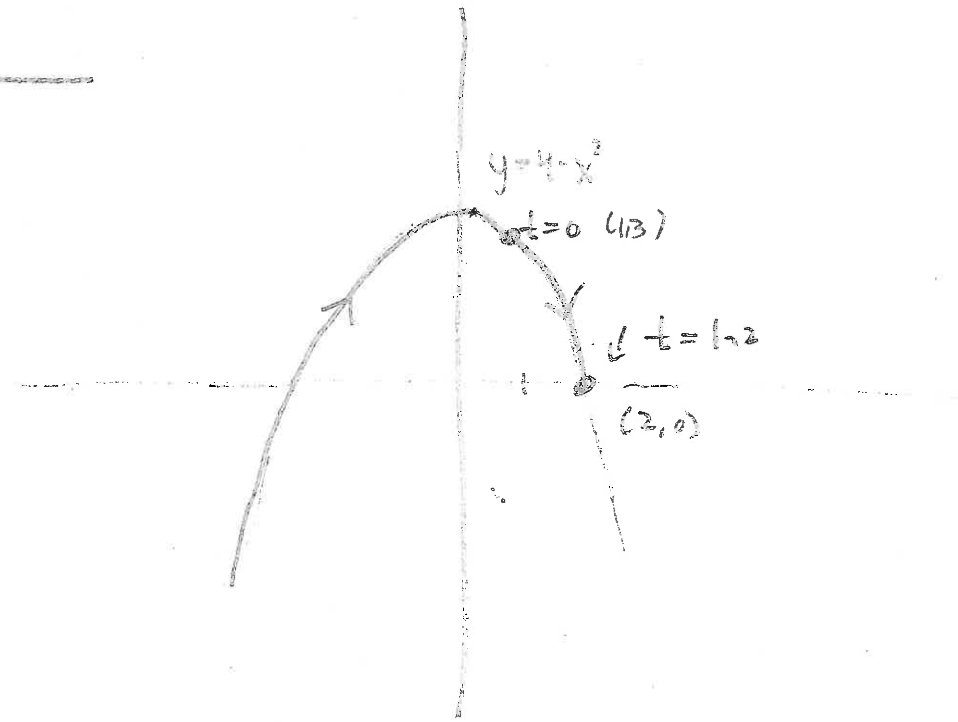
(a) Eliminate the parameter t to obtain an equation in x and y.

$(e^{2t}) = (e^t)^2$ so $y = 4 - (e^t)^2 = 4 - x^2$

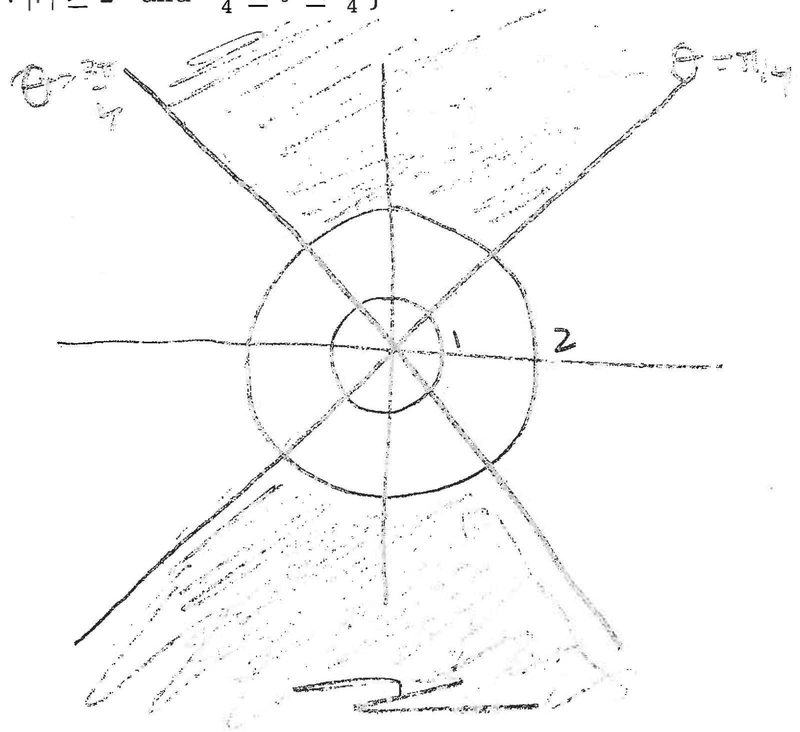
(b) Sketch the curve and indicate the positive orientation.

t	$x = e^t$	$y = 4 - x^2$
$\ln 2$	2	0
0	1	3

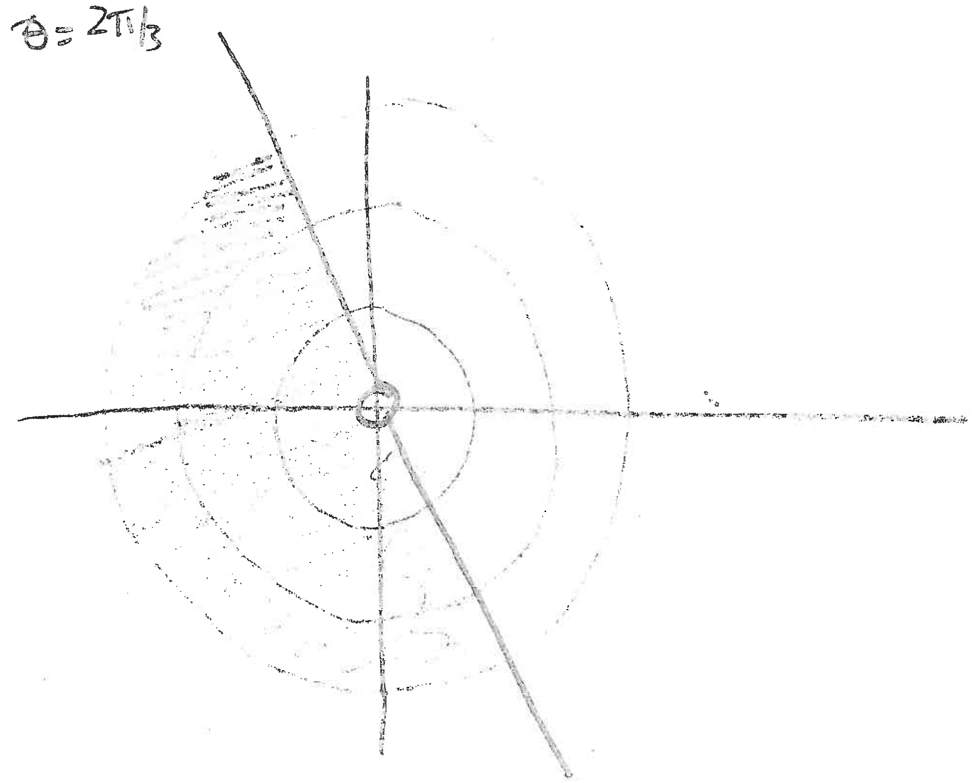


7. (6 points) Sketch each of the following sets of points in the polar plane. (Use separate sketches for each set.)

(a) $\{(r,\theta) : |r| \geq 2 \text{ and } \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\}$

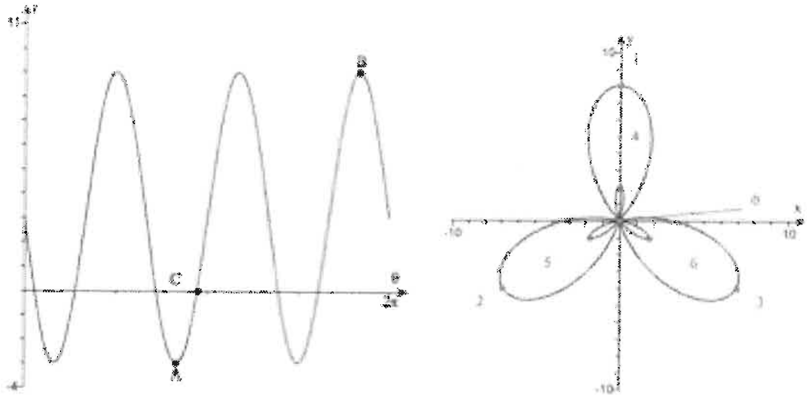


(b) $\{(r,\theta) : 0 < r \leq 3 \text{ and } \frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{3}\}$



8. (8 points) Polar Curves

(a) A Cartesian and a polar graph are given. Identify the points on the polar graph that correspond to the points shown on the Cartesian graph.

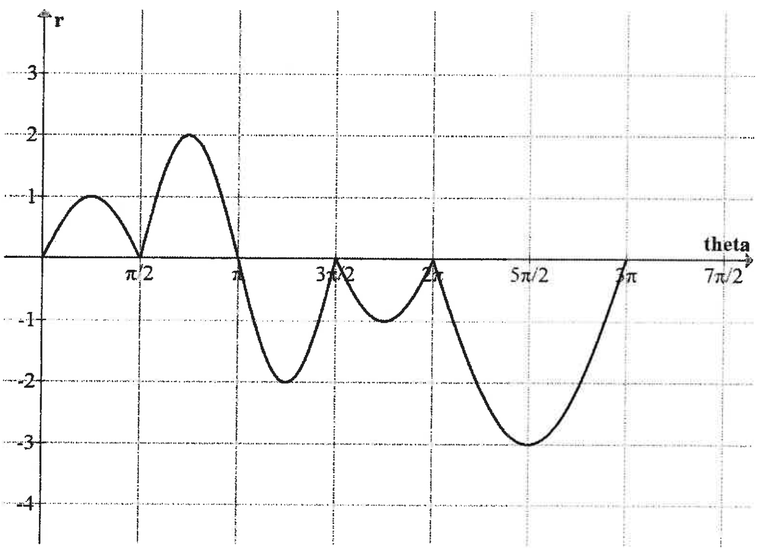


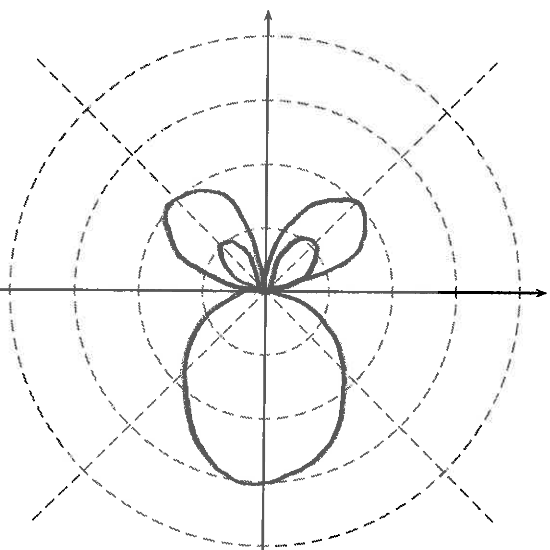
i. Point A on the Cartesian graph corresponds to point 6 on the polar graph.

ii. Point C on the Cartesian graph corresponds to point 0 on the polar graph.

iii. Point B on the Cartesian graph corresponds to point 3 on the polar graph.

(b) The following is the graph of a polar curve drawn in the Cartesian plane. Draw the curve in the polar plane next to it..





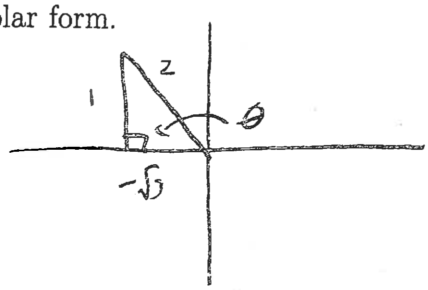
9. (12 points) Complex numbers

(a) i. Write $(-\sqrt{3}+i)$ in polar form.

$(-\sqrt{3}, 1)$

$\cos\theta = -\sqrt{3}/2$

$\sin\theta = 1/2$



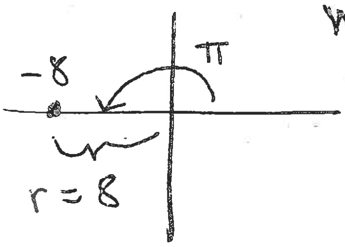
$z = 2e^{i(5\pi/6)}$

ii. Find $(-\sqrt{3}+i)^6$ and write your answer in rectangular form.

$\left(2e^{i(5\pi/6)}\right)^6 = 2^6 e^{i5\pi} = 64\, e^{i\pi} = 64(\cos\pi + i\sin\pi)$

$= 64(-1) = -64$

(b) i. Find the cube roots of $w = -8$ and write your answers in rectangular form.



$W = 8e^{i\pi}$

3 roots $2\pi/3$ apart. First θ is $\pi/3$

radius $= \sqrt[3]{8} = 2$

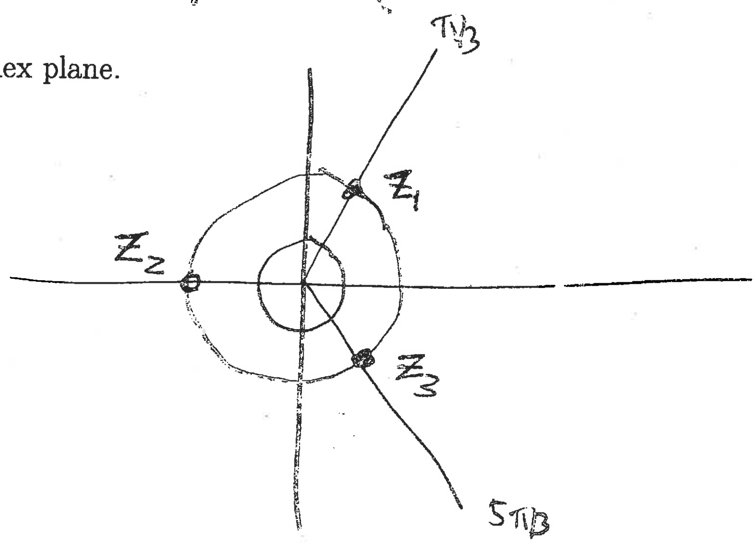
$z_1 = 2e^{i(\pi/3)} = 2[\cos\pi/3 + i\sin\pi/3] = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = 1 + i\sqrt{3}$

$z_2 = 2e^{i\pi} = 2[\cos\pi + i\sin\pi] = 2(-1+0) = -2$

$z_3 = 2e^{i(5\pi/3)} = 2[\cos 5\pi/3 + i\sin 5\pi/3]$

$= 2[1/2 - i\sqrt{3}/2] = 1 - i\sqrt{3}$

ii. Plot the roots in the complex plane.



You may use any of the following for the rest of the exam.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

10. (5 points) Evaluate the following limit using series:

$$\lim_{x\to 0} \frac{2\cos 2x - 2 + 4x^2}{2x^4} = \lim_{x\to 0} \frac{2\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right] - 2 + 4x^2}{2x^4}$$

$$= \lim_{x\to 0} \frac{2\left[1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \cdots\right] - 2 + 4x^2}{2x^4}$$

$$= \lim_{x\to 0} \frac{(\cancel{2} - \cancel{4x^2} + \frac{32x^4}{4!} - \frac{128x^6}{6!} + \cdots) - \cancel{2} + \cancel{4x^2}}{2x^4}$$

$$= \lim_{x\to 0} \frac{16}{4!} - \underbrace{\frac{64x^2}{6!} + \cdots}_{\text{all} \to 0} = \frac{16}{4!} = \frac{16}{24} = \boxed{\frac{2}{3}}$$

11. (6 points) Identify each of the functions represented by the following power series.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{6k+3}}{(2k+1)!}$ $\displaystyle = \sum_{k=0}^{\infty} \frac{(-1)(-1)^k (x^3)^{2k+1}}{(2k+1)!} = -\sum_{k=0}^{\infty} \frac{(-1)^k (x^3)^{2k+1}}{(2k+1)!}$

$$= -\sin(x^3)$$

(b) $\displaystyle\sum_{k=1}^{\infty} 2^k \cdot k \cdot x^{k-1}$ $\displaystyle = \left(\sum_{k=0}^{\infty} 2^k x^k\right)' = \left(\sum_{k=0}^{\infty} (2x)^k\right)' = \left(\frac{1}{1-2x}\right)'$

$$= \left((1-2x)^{-1}\right)' = -(1-2x)^{-2}(-2) = \boxed{\frac{2}{(1-2x)^2}}$$

You may use any of the following for the rest of the exam.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^{k+1}x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

12. (8 points) Use a Taylor series to approximate the following definite integral. Retain as many terms as needed to ensure the error is less than $\frac{1}{5000}$. Justify and simplify your answer.

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\,dx = \int_0^{0.1} \frac{x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots}{x}\,dx$$

$$= \int_0^{0.1} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right) dx$$

$$= \left[x - \frac{x^2}{2\cdot 2} + \frac{x^3}{3\cdot 3} - \frac{x^4}{4\cdot 4} + \frac{x^5}{5\cdot 5} - \cdots\right]_0^{1/10}$$

$$S = \frac{1}{10} - \frac{1}{4\cdot 10^2} + \frac{1}{9\cdot 10^3} - \frac{1}{16\cdot 10^4} + \frac{1}{25\cdot 10^5} \cdots \Big\} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\cdot 10^k}$$

This is a Convergent A.S.

$$\frac{1}{400} \qquad \frac{1}{9000} < \frac{1}{5000}$$

① $\lim_{k\to\infty} \frac{1}{k\cdot 10^k} = 0$ ✓

② $\frac{1}{k\cdot 10^k} > 0$ ✓

③ $\frac{1}{(k+1)10^{k+1}} \le \frac{1}{k\, 10^k}$, $k \ge 1$

So $|S - S_2| \le \frac{1}{9000} < \frac{1}{5000}$

$$S \approx \frac{1}{10} - \frac{1}{400} = \frac{40-1}{400} = \boxed{\frac{39}{400}}$$