Math 12 Final Exam Solutions Fall 2011

① (18) a) $\int \sin^2(x/2)\,dx = \frac{1}{2}\int(1-\cos x)\,dx = \frac{1}{2}(x-\sin x)+C$

b) $\int_0^1 \frac{x+7}{x^2-x-2}\,dx = \int_0^1 \frac{x+7}{(x-2)(x+1)}\,dx = \int_0^1 \frac{3\,dx}{x-2} - \int_0^1 \frac{2\,dx}{x+1} = 3\ln|x-2|\Big|_0^1 - 2\ln|x+1|\Big|_0^1$

$= 3\left[\ln|-1| - \ln|-2|\right] - 2\left[\ln|2| - \ln|1|\right]$ (with $\ln|-1| = 0$ and $\ln|1| = 0$)

$= -3\ln 2 - 2\ln 2 = \boxed{-5\ln 2}$

PF's: $\frac{x+7}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$

⊛ $x+7 = A(x+1) + B(x-2)$

$x = -1 \quad 6 = B(-3) \Rightarrow B = -2$

$x = 2 \quad 9 = A(3) \Rightarrow A = 3$

c) $\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\cdot\ln x - \frac{2}{3}\int \frac{x^{3/2}}{x}\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx$

PARTS: $u = \ln x \quad dv = \sqrt{x}\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{2}{3}x^{3/2}$

$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\left(\frac{2}{3}\right)x^{3/2} + C$

$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$

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② (18) a) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ Integral Test: Let $f(x) = \frac{1}{x\ln x}$, $x \geq 2$. Then f is continuous and positive and

$f'(x) = -\left((x\ln x)^{-2}\left(\ln x + x\cdot\frac{1}{x}\right)\right) < 0$ for $x \geq 2$ so $f(x)$ is decreasing for $x \geq 2$.

$\quad - \qquad + \qquad\qquad +$

$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty} \ln|\ln x|\Big|_2^t = \lim_{t\to\infty}\left[\underset{\downarrow\atop\infty}{\ln|\ln t|} - \ln|\ln 2|\right]$ ∴ The integral diverges

$\int \frac{dx}{x\ln x} \qquad u = \ln x \quad du = \frac{1}{x}\,dx$

$\int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$

so by the integral test, the series also diverges.

b) $\sum_{n=1}^{\infty} \frac{3n^2}{8^n}$ RATIO TEST Let $r = \lim_{n\to\infty} \frac{3(n+1)^2}{8^{n+1}}\cdot\frac{8^n}{3n^2} = \lim_{n\to\infty}\underset{\downarrow\atop 1}{\left(\frac{n+1}{n}\right)^2}\cdot\frac{1}{8} = \frac{1}{8} < 1 \rightarrow$ series converges

c) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3n^3+10}$. Since $\frac{n^2}{3n^3+10} > 0$ for all $n \geq 1$, this is an alternating series.

A.S Test. ① $\lim_{n\to\infty}\frac{n^2}{3n^3+10} = \lim_{n\to\infty}\frac{1/n}{3+10/n^2} = \frac{0}{3} = 0$ ✓

② We will show $\left\{\frac{n^2}{3n^3+10}\right\}$ is eventually decreasing

Let $f(x) = \frac{x^2}{3x^3+10}$

$f'(x) = \frac{2x(3x^3+10) - x^2(6x^2)}{(3x^3+10)^2}$

$= \frac{5x^4+20x-6x^4}{(3x^3+10)^2} = \frac{20x-6x^4}{(3x^3+10)^2} = \frac{2x(10-3x^3)}{(3x^3+10)^2}$

Since $f'(x) < 0$ for all $x \geq 2$ f is eventually decreasing so our sequence of terms is eventually decreasing. ∴ Σ passes A.S.T and converges.

$\boxed{3}$ (8) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{4^n \sqrt{n}}$ RATFACE! (1) Let $\displaystyle r = \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{4^{n+1}\sqrt{n+1}} \cdot \frac{4^n \sqrt{n}}{(x+2)^n} \right| = |x+2| \lim_{n\to\infty} \frac{1}{4} \cdot \sqrt{\frac{n}{n+1}} = \frac{|x+2|}{4}$

center $a=-2$

(2) AbC: $\frac{|x+2|}{4} < 1 \iff |x+2| < 4 \iff -4 < x+2 < 4$

$-6 < x < 2$

ROC = 4

(3) Endpts: (i) $x=-6$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{4^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ This is a divergent p-Σ $p=\frac{1}{2} \leq 1$

(ii) $x=2$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ A.S. Test. Since $\frac{1}{\sqrt{n}} > 0$ for $n \geq 1$, this is an alternating series.

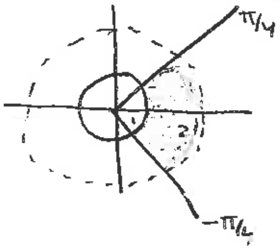
① $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

② $\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ for all $n \geq 1$ so $\left\{\frac{1}{\sqrt{n}}\right\}$ is a decreasing sequence ✓

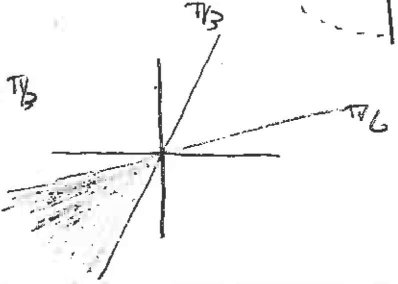
∴ This series passes the A.S.T + converges.

answer: ROC = 4 IOC = $(-6, 2]$

$\boxed{4}$ (4) a) $\{(r,\theta): 1 \leq r < 2$ and $-\pi/4 \leq \theta \leq \pi/4\}$



b) $\{(r,\theta): r \leq 0$ and $\pi/6 \leq \theta \leq \pi/3\}$



$\boxed{5}$ (8)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	xe^x	e
1	$e^x + xe^x$	$e+e=2e$
2	$\underbrace{e^x + e^x}+xe^x$ $2e^x + xe^x$	$2e+e=3e$
3	$\underbrace{2e^x + e^x}+xe^x$ $3e^x+xe^x$	$3e+e=4e$
⋮		
n	ne^x+xe^x	$ne+e=(n+1)e$

$\displaystyle T\Sigma = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)e(x-1)^n}{n!}$

RATFACE $\displaystyle r = \lim_{n\to\infty} \left| \frac{(n+2)e(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(n+1)e(x-1)^n} \right|$

$\displaystyle = |x-1| \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{n+1}{n^2+2n+1}$

$\displaystyle = \lim_{n\to\infty} \frac{1/n + 2/n^2}{1 + 2/n + 1/n^2} = \frac{0}{1} = 0 < 1$ always

∴ IOC $= (-\infty, \infty)$

$\boxed{6}$ (6)

$x = 3 - 2\cos t$

$y = 4\sin^2 t$, $\pi \leq t \leq 2\pi$

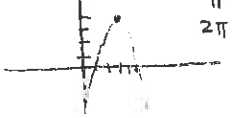
$\frac{x-3}{2} = \cos t$, $\frac{y}{4} = \sin^2 t$

$\left(\frac{x-3}{2}\right)^2 + \frac{y}{4} = 1$

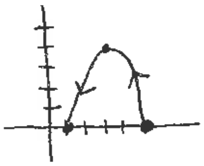
$\frac{(x-3)^2}{4} + \frac{y}{4} = 1$

$(x-3)^2 + y = 4$

$Y = -(x-3)^2+4$



t	x	y
π	5	0
2π	1	0



$\boxed{7}$ (12) a) $f(x) = \frac{x^4}{6+x} = x^4\left(\frac{1}{6+x}\right) = x^4\left(\frac{1}{6(1+x/6)}\right) = x^4\left(\frac{1}{6(1-(-x/6))}\right) = \frac{x^4}{6}\left(\frac{1}{1-(-x/6)}\right)$

$= \frac{x^4}{6}\sum_{n=0}^{\infty}\left(\frac{-x}{6}\right)^n = \sum_{n=0}^{\infty}\frac{x^4}{6}\frac{(-1)^n x^n}{6^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+4}}{6^{n+1}} \quad \text{ROC} = 6}$

ROC: $\left|\frac{x}{6}\right| < 1$

$|x| < 6$

$R_x = 6$

b) $f(x) = \ln(1-2x)$ (i) $(\ln(1-2x))' = \frac{-2}{1-2x} = -2\left(\frac{1}{1-2x}\right) = -2\sum_{n=0}^{\infty}(2x)^n = -\sum_{n=0}^{\infty}2^{n+1}x^n$ $|x|<1$ ROC = 1

$\therefore f(x) = \ln(1-2x) = C + \int\frac{-2\,dx}{1-2x} = C - \int\sum_{n=0}^{\infty}2^{n+1}x^n$

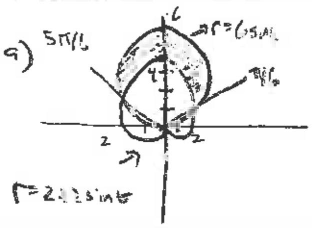
$\ln(1-2x) = C - \sum_{n=0}^{\infty}\frac{2^{n+1}x^{n+1}}{n+1}$ $\ln(1-2x) = -\sum_{n=0}^{\infty}\frac{2^{n+1}x^{n+1}}{n+1}$

Solve for C: Let x=0 $\underset{0}{\ln(1)} = C + 0 \Rightarrow C = 0$

$\boxed{8}$ (4) a) $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^{4k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{(-1)(-1)^k(x^2)^{2k}}{(2k)!} = -\cos(x^2)$

b) $\sum_{k=0}^{\infty}\frac{(-1)^k x^{k+3}}{5k!} = \frac{1}{5}\sum_{k=0}^{\infty}\frac{(-1)^k x^k x^3}{k!} = \frac{x^3}{5}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!} = \frac{x^3}{5}e^{-x}$

$\boxed{9}$ (6) a)



b) $6\sin\theta = 2+2\sin\theta$

$4\sin\theta = 2$

$\sin\theta = 1/2$

$\theta = \pi/6, 5\pi/6$

$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[(6\sin\theta)^2 - (2+2\sin\theta)^2\right]d\theta$

$\boxed{10}$ (14) a) $f(x) = \ln x$

i)

n	$f^{(n)}(x)$
0	$\ln x$
1	$1/x$
2	$-1/x^2$
3	$2/x^3$

$R_2(x) = \frac{f^{(3)}(c)(x-1)^3}{3!}$ where c is between x and 1

$\boxed{R_2(x) = \frac{2(x-1)^3}{c^3\cdot 3!} \quad \text{where c is between x and 1}}$

(ii) $R_2(\tfrac{1}{2}) = \frac{2(\frac{1}{2}-1)^3}{c^3\,3!} = \frac{2}{c^3\cdot 3!\,(-2)^3} = \frac{2}{c^3\cdot 6\cdot(-8)} = \frac{-1}{c^3\,24}$ $\frac{1}{2} < c < 1$

So $|R_2(\tfrac{1}{2})| = \left|\frac{-1}{24c^3}\right| \le \frac{1}{24(\frac{1}{2})^3} = \frac{8}{24} = \boxed{\frac{1}{3}}$

(b)

n	$f^{(n)}(x)$	$f^{(n)}(\pi/3)$
0	$\sin 2x$	$\sin\frac{2\pi}{3} = +\sqrt{3}/2$ $\frac{1}{2}$
1	$2\cos 2x$	$2\cos\frac{2\pi}{3} = 2(-\frac{1}{2}) = -1$ $-\sqrt{3}$
2	$-4\sin 2x$	$-4(\sqrt{3}/2) = -2\sqrt{3}$ -2
3	$-8\cos 2x$	$-8(-\frac{1}{2}) = 4$ $4\sqrt{3}$

$T_3(x) = f(\pi/3) + f'(\pi/3)(x-\pi/3) + \frac{f''(\pi/3)}{2!}(x-\pi/3)^2 + \frac{f^{(3)}(\pi/3)}{3!}(x-\pi/3)^3$

$T_3(x) = \frac{\sqrt{3}}{2} - (x-\pi/3) - \frac{2\sqrt{3}}{2}(x-\pi/3)^2 + \frac{4}{6}(x-\pi/3)^3$

$\boxed{T_3(x) = \frac{\sqrt{3}}{2} - (x-\pi/3) - \sqrt{3}(x-\pi/3)^2 + \frac{2}{3}(x-\pi/3)^3}$