EXAM 2 SOLS

(1) (12 pts) **True or false.** Indicate your answer by circling T or F.

(a) If a series $\sum_{k=1}^{\infty} a_k$ sums to 15, then the sequence $\{a_k\}$ converges to 15. T **F**

(b) If $\sum_{k=1}^{\infty} a_k$ converges, and $0 < a_k \le b_k$ for all $k \ge 1$, then $\sum_{k=1}^{\infty} b_k$ converges. T **F**

(c) The sequence $\{\sin n\}$ converges because it is bounded. T **F**

(d) For every p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, the ratio test is inconclusive. **T** F

(e) The sequence $\left\{\frac{(-1)^n+3}{\sqrt{n}}\right\}$ converges to 0 by the Squeeze Theorem. **T** F

(f) $-\frac{3}{5}+\frac{9}{25}-\frac{27}{125}+\frac{81}{625}-\cdots = -\frac{3}{8}$ **T** F

(2) (8 pts) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$, and suppose you know it converges.

Estimate the sum of the series with an absolute error less than $\frac{1}{500}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} = -\frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots$$

$$= -\frac{1}{2} + \frac{1}{24} - \underbrace{\frac{1}{720}} + \cdots$$

the magnitude of the remainder is less than the first neglected term

$(R_n \le a_{n+1})$

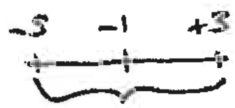
So since $\frac{1}{720} < \frac{1}{500}$, the partial sum $\boxed{-\frac{1}{2}+\frac{1}{24}}$ estimates the series with absolute error less than $\frac{1}{500}$.

$2! = 2$, $4! = 4\cdot3\cdot2 = 24$, $6! = 6\cdot5\cdot4\cdot3\cdot2 = 30\cdot24 = 720$

(3) (15 pts) **Multiple choice.** Circle ALL correct answers.

(a) Suppose a power series $\sum_{n=0}^{\infty} c_n(x+1)^n$ has radius of convergence 4.

Then the series ...



(i) converges at $x = -2$;

(ii) converges at $x = 0$;

(iii) converges at $x = 2$;

~~(iv)~~ converges at $x = 4$.

(b) The series $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ ...

~~(i)~~ converges by the integral test;

~~(ii)~~ converges by the ratio test;

(iii) diverges by the ratio test;

~~(iv)~~ is a divergent geometric series;

~~(v)~~ converges by the divergence test.

(c) A sequence has the recursive formula $a_{n+1} = a_n + 3$ and initial value $a_0 = 2$.

An explicit formula for the nth term is ...

~~(i)~~ $a_n = n + 2, \quad n \geq 0$;

~~(ii)~~ $a_n = 2n + 2, \quad n \geq 0$;

(iii) $a_n = 3n + 2, \quad n \geq 0$;

~~(iv)~~ $a_n = 3n + 1, \quad n \geq 0$.

(d) The series $\sum_{k=1}^{\infty} \left(\frac{5}{k+2}\right)^k$ ...

~~(i)~~ diverges by the root test;

(ii) converges by the root test;

~~(iii)~~ makes the root test inconclusive;

~~(iv)~~ is a geometric series;

~~(v)~~ diverges by the divergence test.

(e) The sequence $\left\{\left(1 - \frac{2}{k}\right)^k\right\}$ ...

~~(i)~~ converges to 0;

(ii) converges to $1/e^2$;

~~(iii)~~ converges to 1;

~~(iv)~~ converges to e^2;

~~(v)~~ diverges.

(4) (15 pts) Determine whether each of the following series converges or diverges. If the series converges, find its sum.

(a) $\sum_{k=1}^{\infty} \ln\left(2 + \frac{1}{k}\right)$

Here, $a_k = \ln\left(2 + \frac{1}{k}\right)$. We have $\lim_{k \to \infty} a_k = \ln 2 > 0$

So the series DIVERGES by the divergence test.

(b) $\sum_{k=1}^{\infty} \left(\frac{1}{k+3} - \frac{1}{k+2}\right)$

This is a telescoping series. Look at the partial sums:

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k+3} - \frac{1}{k+2}\right) = \left(\cancel{\frac{1}{4}} - \frac{1}{3}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{5}}\right)$$

$$+ \dots + \left(\cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+1}}\right) + \left(\frac{1}{n+3} - \cancel{\frac{1}{n+2}}\right)$$

$$= \frac{1}{n+3} - \frac{1}{3}. \quad \text{(all other terms cancel)}$$

We have $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(\frac{1}{n+3} - \frac{1}{3}\right) = -\frac{1}{3}$, so the series sums to $-\frac{1}{3}$.

(5) (16 pts) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{k+1}{k^2}$

This is an alternating series.

First check for absolute convergence: does $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ converge?

Well, if $a_k = \frac{k+1}{k^2}$, I can guess that this grows about as fast as a series with terms $b_k = \frac{1}{k}$.

LIMIT COMPARISON TEST: (hypotheses: $a_k, b_k \geq 0$ ✓)

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{(k+1)/k^2}{1/k} = \lim_{k\to\infty} \frac{k+1}{k} = \underbrace{\lim_{k\to\infty} \frac{1+1/k}{1}}_{} = 1$$

(slightly anal step!)

So $\sum a_k$ and $\sum b_k$ have the "same fate" — they either both converge or both diverge.

Since $\sum \frac{1}{k}$ diverges (harmonic series), $\sum \frac{k+1}{k^2}$ does as well.

So ours is <u>not</u> absolutely convergent.

Next use Alternating Series Test to check for conditional convergence.

Again, $a_k = \frac{k+1}{k^2}$. Check hypotheses: • $a_k \geq 0$ ✓

• $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+1}{k^2} = \lim_{k\to\infty} \frac{\frac{1}{k} + \frac{1}{k^2}}{1} = 0$ ✓

• eventually decreasing: let $f(x) = \frac{x+1}{x^2} = \frac{1}{x} + \frac{1}{x^2} = x^{-1} + x^{-2}$.

Then $f'(x) = \frac{-1}{x^2} + \frac{-2}{x^3}$ which is clearly negative when x is positive.

$f'(x) < 0 \Rightarrow f$ is <u>decreasing</u> ✓

So, series is

CONDITIONALLY CONVERGENT

(b) $\displaystyle\sum_{k=1}^{\infty} k \left(-\frac{1}{2}\right)^k$

First, check absolute convergence: does $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ converge?

RATIO TEST (hypotheses: $a_k \geq 0$ ✓)

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \lim_{k\to\infty} \frac{k+1}{2k} = \lim_{k\to\infty} \frac{1+1/k}{2} = \frac{1}{2}$$

Since $r<1$, this series converges,

so original series CONVERGES ABSOLUTELY.

(6) (8 pts) Recall that $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$.

Find a power series representation for $f(x) = \dfrac{x}{1+\frac{x^2}{4}}$ and give its radius of convergence.

$$f(x) = \frac{x}{1+(x^2/4)} = x\left(\frac{1}{1-\left(\frac{-x^2}{4}\right)}\right)$$

$$= x \sum_{n=0}^{\infty} \left(\frac{-x^2}{4}\right)^n = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^n}$$

if $\left|\frac{-x^2}{4}\right| < 1$

$\Updownarrow$

$x^2 < 4$ $\quad (|x|<2)$

$\Updownarrow$

$-2 < x < 2$. So radius of convergence is 2.

(7) (8 pts) Find the radius of convergence and interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(-1)^k \cdot (x-3)^k}{k \cdot 2^k}$.

We'll use the RATIO TEST FOR ABSOLUTE CONVERGENCE.

$$r = \lim_{k\to\infty} \frac{|x-3|^{k+1} / (k+1)\cdot 2^{k+1}}{|x-3|^k / k\cdot 2^k} = \lim_{k\to\infty} \cancel{\frac{k}{k+1}}^{\,1} \cdot \frac{1}{2} \cdot |x-3| = \frac{|x-3|}{2}.$$

We get convergence when $r<1$, i.e.,

$$\frac{|x-3|}{2} < 1 \Leftrightarrow |x-3| < 2 \Leftrightarrow +1 < x < 5$$

Now, check endpoints:

$(x=+1)$: $\sum \frac{(-1)^k(-2)^k}{k\cdot 2^k} = \sum \frac{1}{k}$ diverges (harmonic)

$(x=5)$: $\sum \frac{(-1)^k 2^k}{k \cdot 2^k} = \sum \frac{(-1)^k}{k}$ converges (alternating harmonic)

So interval of convergence is $(1,5]$.

radius of convergence is 2.

(8) (8 pts) Starting with the power series $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(x-2)^k}{3^k}$ ...

(a) find a power series representation for $f''(x)$.

$$f'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \cdot k\,(x-2)^{k-1}}{3^k}$$

$$f''(x) = \sum_{k=2}^{\infty} \frac{(-1)^k (k)(k-1)(x-2)^{k-2}}{3^k}$$

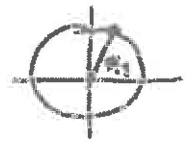
(b) find a power series representation for $\int f(x)\,dx$.

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \frac{1}{k+1}\,(x-2)^{k+1}}{3^k} + C$$

(9) (10 pts)

(a) For $f(x) = \sin(2x)$, find the fourth-degree Taylor polynomial $p_4(x)$ centered at $a = \frac{\pi}{6}$.

Generally, $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.



$f(x) = \sin 2x$; $f(\frac{\pi}{6}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f'(x) = 2\cos 2x$; $f'(\frac{\pi}{6}) = 2\cos\frac{\pi}{3} = 2(\frac{1}{2}) = 1$

$f''(x) = -4\sin 2x$; $f''(\frac{\pi}{6}) = -4(\frac{\sqrt{3}}{2}) = -2\sqrt{3}$

$f'''(x) = -8\cos 2x$; $f'''(\frac{\pi}{6}) = -8(\frac{1}{2}) = -4$

$f''''(x) = +16\sin 2x$; $f''''(\frac{\pi}{6}) = 16(\frac{\sqrt{3}}{2}) = 8\sqrt{3}$

$$p_4(x) = \frac{\sqrt{3}/2}{1}\left(x-\frac{\pi}{6}\right)^0 + \frac{1}{1}\left(x-\frac{\pi}{6}\right)^1 + \frac{-2\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^2 + \frac{-4}{6}\left(x-\frac{\pi}{6}\right)^3$$
$$+ \frac{8\sqrt{3}}{24}\left(x-\frac{\pi}{6}\right)^4 = \boxed{\frac{\sqrt{3}}{2} + \left(x-\frac{\pi}{6}\right) - \sqrt{3}\left(x-\frac{\pi}{6}\right)^2 - \frac{2}{3}\left(x-\frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{3}\left(x-\frac{\pi}{6}\right)^4}$$

(b) What is the quadratic polynomial that best approximates $\sin(2x)$ near $x = \frac{\pi}{6}$?

That is $p_2(x)$, which is simply

$$\frac{\sqrt{3}}{2} + \left(x-\frac{\pi}{6}\right) - \sqrt{3}\left(x-\frac{\pi}{6}\right)^2.$$

1	2	3	4	5	6	7	8	9	TOTAL
[12]	[8]	[15]	[15]	[16]	[8]	[8]	[8]	[10]	[100]