KEY

Math 34
Calculus II

TUFTS UNIVERSITY
Department of Mathematics
Exam 2

November 9, 2015
All sections

No books, notes, **or calculators**. **TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You are not allowed to quote results about growth rates. You are required to **sign** your exam on the last page. With your signature, you pledge that you have neither given nor received assistance on this exam. There is a blank page at the end of the exam. Be sure to indicate any work on it that you want graded.

Problem	Point Value	Points
1	15	
2	16	
3 (a)	8	
3 (b)	8	
3 (c)	8	
4	18	
5	7	
6	10	
7	10	
	100	

1. (15 points) Determine whether each of the following sequences converges or diverges. If the sequence converges, find the limit.

(a) $a_n = \dfrac{n^2+1}{(n+1)^2} = \dfrac{n^2+1}{n^2+2n+1}$

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+2n+1} = \lim_{n\to\infty} \frac{1+\overset{\to 0}{1/n^2}}{1+\underset{\downarrow\atop 0}{2/n}+\underset{\downarrow\atop 0}{1/n^2}} = 1$$

Seq. c's to 1

(b) $a_n = \dfrac{\cos(n)}{n}$

Squeeze Thm

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}, \quad n \geq 1$$

$\to 0 \leftarrow$

Seq. c's to 0 by the Squeeze Theorem

(c) $a_n = n^{2/n}$

$$\lim_{n\to\infty} n^{2/n}$$

Form ∞^0

(i) $\displaystyle\lim_{n\to\infty} \ln n^{2/n} = \lim_{n\to\infty} \underset{\downarrow\atop 0}{\frac{2}{n}}\, \underset{\downarrow\atop \infty}{\ln n} = \lim_{n\to\infty} \frac{2\ln n}{n}\begin{matrix}\to\infty\\ \to\infty\end{matrix} =$

$$\lim_{x\to\infty} \frac{2\ln x}{x}\begin{matrix}\to\infty\\ \to\infty\end{matrix} \overset{(L)}{=} \lim_{x\to\infty} \frac{2}{x} = 0$$

(ii) $\displaystyle\lim_{n\to\infty} n^{2/n} = e^0 = 1$

2. (16 points) Determine the convergence or divergence of the following series. Justify your answer. State and check hypotheses of any test, rules or theorems you use. *If the series converges, find its sum.*

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{4^{k-1}}$ $= 3 + \frac{3^2}{4} + \frac{3^3}{4^2} + \frac{3^4}{4^3} + \cdots$

Geometric $r = 3/4$ Since $|r| = \frac{3}{4} < 1$,

Σ c's to the sum $S = \frac{a}{1-r} = \frac{3}{1-3/4} = \frac{3}{1/4} = 12$

(b) $\displaystyle\sum_{k=1}^{\infty} \left(\sqrt{k+1} - \sqrt{k}\right)$ Telescoping

$k=1 \quad k=2 \quad k=3 \quad k=n-1 \quad k=n$

$S_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{n} - \sqrt{n-1}) + (\sqrt{n+1} - \sqrt{n})$

$= (\cancel{\sqrt{2}} + \cancel{\sqrt{3}} + \cancel{\sqrt{4}} + \cdots + \cancel{\sqrt{n}} + \sqrt{n+1}) - (\sqrt{1} + \cancel{\sqrt{2}} + \cancel{\sqrt{3}} + \cancel{\sqrt{4}} + \cdots + \cancel{\sqrt{n-1}} + \cancel{\sqrt{n}})$

$S_n = \sqrt{n+1} - \sqrt{1}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n+1} - 1 \to \infty$ $\therefore \Sigma$ D's by the definition of series convergence/divergence.

3. (24 points) Determine the convergence or divergence of the series below. Justify your answer. State and check hypotheses of any test, rules or theorems you use. (8 point each)

(a) $\sum_{k=1}^{\infty} \frac{3+\tan^{-1}(k)}{e^k}$ $\sim$ $\underbrace{\sum_{k\geq 1}^{\infty} \frac{1}{e^k}}$

convergent geometric series : $|r| = |\frac{1}{e}| < 1$

option 1 OCT

$$0 < \frac{3+\tan^{-1}k}{e^k} \leq \frac{5}{e^k} \quad , k \geq 1$$

Since $\sum \frac{5}{e^k}$ is still a convergent geom. $\sum$ ($r = 1/e$ and $|\frac{1}{e}| < 1$) our $\sum$ c's by OCT

option 2 LCT

$$L = \lim_{k\to\infty} \frac{3+\tan^{-1}k}{e^k} \cdot \frac{e^k}{1} = \lim_{k\to\infty} 3+\tan^{-1}k = 3+\pi/2$$

Since $0 < 3+\pi/2 < +\infty$ our $\sum$ c's by LCT

(b) $\sum_{k=1}^{\infty} \frac{5^{k+1}}{k!}$

RATIO Test

$$r = \lim_{k\to\infty} \frac{5^{(k+1)+1}}{(k+1)!} \cdot \frac{k!}{5^{k+1}} = \lim_{k\to\infty} \frac{5^{k+2}}{5^{k+1}} \cdot \frac{k!}{(k+1)\cdot k!}$$

$$= 5 \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$$

$\sum$ c's by the RATIO Test.

Determine the convergence or divergence of the series below. Justify your answer. State and check hypotheses of any test, rules or theorems you use.

(c) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+2}{5k-1}\right)^k$ Root Test

$$\rho = \lim_{k\to\infty} \left(\left(\frac{k+2}{5k-1}\right)^k\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{k+2}{5k-1} =$$

$$\lim_{k\to\infty} \frac{1 + 2/k \to 0}{5 - 1/k \to 0} = \frac{1}{5} < 1. \quad \Sigma \text{ c's by the Root Test.}$$

4. (18 points) Determine whether each of the following series converges absolutely, conditionally, or diverges. Justify your answer. State and check hypotheses of any test, rules or theorems you use.

(a) $\sum_{k=1}^{\infty}(-1)^k \frac{k}{7k+1}$

DN Test : (i) $\lim_{k \to \infty} \frac{k}{7k+1} = \lim_{k \to \infty} \frac{1}{7 + (1/k)_{\to 0}} = \frac{1}{7} \neq 0$

(ii) $\lim_{k \to \infty} \frac{(-1)^k k}{7k+1} \neq 0$ (oscillates odd terms $\Rightarrow -1/7$, even terms $\Rightarrow 1/7$)

$\therefore \sum$ D's by the Divergence Test.

Determine whether the following series converges absolutely, conditionally, or diverges. Justify your answer. State and check hypotheses of any test, rules or theorems you use.

(b) $\sum_{k=1}^{\infty}(-1)^k \frac{\ln k}{k}$

$\boxed{1}$ AbC? $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

OPTION 1 OCT $\sum_{k=1}^{\infty} \frac{\ln k}{k} \sim \sum_{k=1}^{\infty} \frac{1}{k}$ Since $0 < \frac{1}{k} \le \frac{\ln k}{k}$, $k \geq 3$

and since $\sum \frac{1}{k}$ is the divergent H-Σ, our Σ is not AbC

OPTION 2 Integral Test Let $f(x) = \frac{\ln x}{x}$, $x \geq 1$.

Then $f(x)$ is continuous and positive, $x>1$. Since $f'(x) = \frac{x(1/x) - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$ for $x>e$,

$f(x)$ decreases for $x>e$.

$I = \int_1^{\infty} \frac{\ln x \, dx}{x} = \lim_{t\to\infty} \int_1^t \frac{\ln x \, dx}{x} = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty}\left[\frac{(\ln t)^2}{2} - \left(\frac{\ln 1}{2}\right)^2\right]$

$\frac{(\ln t)^2}{2} \to \infty$, $\left(\frac{\ln 1}{2}\right)^2 = 0$

$\left[\begin{array}{l} \int \frac{\ln x \, dx}{x} = \int u\,du = \frac{u^2}{2} + C \\ u = \ln x, \; du = dx/x \quad = \frac{(\ln x)^2}{2} + C \end{array}\right]$

Since I Diverges, our Σ is not AbC.

$\boxed{2}$ CC? $\sum \frac{(-1)^k \ln k}{k}$

A.S.T.

① $\lim_{k\to\infty} \frac{\ln k}{k} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{Ⓛ}{=} \lim_{x\to\infty} \frac{1}{x} = 0$ ✓

② $\frac{\ln k}{k} > 0$ for $k \geq 2$

③ To show $\left\{\frac{\ln k}{k}\right\}$ decreases eventually:

Let $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{x(1/x) - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$ for $x > e$

$\therefore \left\{\frac{\ln k}{k}\right\}$ decreases for $k \geq 3$ ✓

Σ passes A.S.T and is Conditionally Convergent (CC)

5. (7 points) The alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1}$ satisfies the conditions of the alternating series theorem. Estimate the value of the alternating series with an absolute error less than 0.1. Justify your work and simplify your answer.

We will use the result $|S - S_n| \leq b_{n+1}$ where

$S = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1}$, $S_n =$ nth partial sum and $b_{n+1} = \left|\frac{(-1)^{n+1}}{(n+1)^2+1}\right| = \frac{1}{(n^2+1)+1}$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1} = -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \frac{1}{17} - \frac{1}{26} + \dots$$

↑

Since this is the first term $< \frac{1}{10}$,

this is our b_{n+1}

So $|\text{error}| < \frac{1}{10}$ means

$$S_n = S_3 = -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{-5+2-1}{10} = \frac{-4}{10} = \frac{-2}{5}$$

(or -0.4)

6. (10 points) Quadratic Approximation

(a) Find the 2nd-order Taylor polynomial $p_2(x)$ for $f(x) = \sqrt{x+1}$ centered at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sqrt{x+1} = \frac{1}{2}(x+1)$	$\frac{1}{2}$
1	$\frac{1}{2}(x+1)^{-1/2}$	$-\frac{1}{2}$
2	$-\frac{1}{4}(x+1)^{-3/2}$	$-\frac{1}{4}$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$p_2(x) = \frac{1}{2} - \frac{1}{2}x - \frac{1/4}{2}x^2$$

$$p_2(x) = \frac{1}{2} - \frac{1}{2}x - \frac{1}{8}x^2$$

(b) Use the polynomial you found to approximate $f(0.1)$. You do not have to simplify your answer.

$$f(0.1) \approx p_2(0.1) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{10}\right) - \frac{1}{8}\left(\frac{1}{10}\right)^2$$

$$= \frac{1}{2} - \frac{1}{20} - \frac{1}{800}$$

7. (10 points) Find the radius of convergence and interval of convergence of the power series. You must show all work to receive full credit.

$$\sum_{k=1}^{\infty} \frac{2^k x^k}{\sqrt{k}}$$ (center is $a=0$)

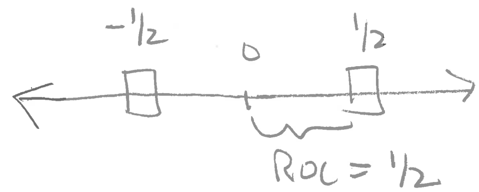
$\boxed{1}$ RATIO: $r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{2^{k+1} x^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{2^k x^k} \right|$

$= 2|x| \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k+1}} = 2|x| \lim_{k\to\infty} \sqrt{\frac{k}{k+1}} = 2|x| \underbrace{\lim_{k\to\infty} \sqrt{\frac{1}{1+1/k}}}_{1} = 2|x|$

$1/k \to 0$

$\boxed{2}$ Solve $2|x| < 1$

$|x| < 1$



$\boxed{3}$ Endpts

(i) $x = -\frac{1}{2}$ $\quad \sum_{k\geq 1}^{\infty} \frac{2^k(-\frac{1}{2})^k}{\sqrt{k}} = \sum_{k\geq 1}^{\infty} \frac{(-\frac{2}{2})^k}{\sqrt{k}} = \sum_{k\geq 1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

A.S.T: 1) $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$ ✓

2) $\frac{1}{\sqrt{k}} > 0$, $k \geq 1$ ✓

3) Since $\frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k}}$, $k \geq 1$, $\{\frac{1}{\sqrt{k}}\}$ decreases

$\therefore$ Σ passes A.S.T

(ii) $x = \frac{1}{2}$ $\quad \sum_{k\geq 1}^{\infty} \frac{2^k(\frac{1}{2})^k}{\sqrt{k}} = \sum_{k\geq 1}^{\infty} \frac{(1)^k}{\sqrt{k}} = \sum_{k\geq 1}^{\infty} \frac{1}{\sqrt{k}}$

This is a divergent p-Σ: $p = 1/2 \leq 1$.

IoC $= [-1/2, 1/2)$ and ROC $= 1/2$