Math 12 Exam II F 2011 Solutions

[1] (10) T F F F F

[2] (14) a) $\lim_{n\to\infty}(3+n)^{1/n}$ Form ∞^0

(i) $\lim_{n\to\infty}\ln(3+n)^{1/n} = \lim_{n\to\infty}\underbrace{\tfrac{1}{n}}_{0}\underbrace{\ln(3+n)}_{\infty} = \lim_{n\to\infty}\frac{\ln(3+n)\to\infty}{n\to\infty}$

$= \lim_{x\to\infty}$ (switch to x) $\frac{\ln(3+x)\to\infty}{x\to\infty} \overset{L}{=} \lim_{x\to\infty}\frac{1}{3+x} = 0$ (ii) $\lim_{n\to\infty}(3+n)^{1/n} = e^0 = 1$

$\boxed{\text{Sequence converges to } 1}$

(b) $\lim_{n\to\infty}\frac{2^n+7^n}{4^n+5^n} = \lim_{n\to\infty}\frac{\frac{2^n}{5^n}+\frac{7^n}{5^n}}{\frac{4^n}{5^n}+\frac{5^n}{5^n}} = \lim_{n\to\infty}\frac{\left(\frac{2}{5}\right)^n+\left(\frac{7}{5}\right)^n}{\left(\frac{4}{5}\right)^n+1}$

$\left(\frac{2}{5}\right)^n \to 0 \ (|\tfrac{2}{5}|<1)$, $\left(\frac{7}{5}\right)^n\to\infty$, $\left(\frac{4}{5}\right)^n \to 0 \ (|\tfrac{4}{5}|<1)$

$|\tfrac{7}{5}|>1 \to \infty$ $\boxed{\text{Sequence diverges}}$

[3] (10) $\frac{dy}{dt} = \frac{e^{2t}+\cos t}{y^2}$, $y(0)=-1$

$y^2\,dy = e^{2t}+\cos t\,dt$

$\frac{y^3}{3} = \frac{1}{2}e^{2t}+\sin t + c$

$y^3 = \frac{3}{2}e^{2t}+3\sin t + c$

$y = \sqrt[3]{\frac{3}{2}e^{2t}+3\sin t + c}$

solve for c: $-1 = \sqrt[3]{\frac{3}{2}(1)+0+c}$

$-1 = \frac{3}{2}+c$ so $c = -\frac{5}{2}$

$\boxed{y = \sqrt[3]{\tfrac{3}{2}e^{2t}+3\sin t - \tfrac{5}{2}}}$

[4] (20) a) $\sum_{k=1}^{\infty}(-1)^k\frac{7^k}{4e^k} = \frac{-7}{4e}+\frac{7^2}{4e^2}-\frac{7^3}{4e^3}+\cdots$ Geometric Series $|r| = \left|-\frac{7}{e}\right| = \frac{7}{e}>1$ $\therefore \sum$ D's

(or use divergence test)

b) $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+2}}-\frac{1}{\sqrt{k+1}}\right)$ $S_n = \left(\frac{1}{\sqrt3}-\frac{1}{\sqrt2}\right)+\left(\frac{1}{\sqrt4}-\frac{1}{\sqrt3}\right)+\left(\frac{1}{\sqrt5}-\frac{1}{\sqrt4}\right)+\cdots+\left(\frac{1}{\sqrt{n+2}}-\frac{1}{\sqrt{n+1}}\right)$

$= \left(\frac{1}{\sqrt3}+\frac{1}{\sqrt4}+\cdots+\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}\right) - \left(\frac{1}{\sqrt2}+\frac{1}{\sqrt3}+\frac{1}{\sqrt4}+\cdots+\frac{1}{\sqrt{n+1}}\right)$

$S_n = \frac{1}{\sqrt{n+2}}-\frac{1}{\sqrt2}$ $\lim_{n\to\infty}\left(\underbrace{\frac{1}{\sqrt{n+2}}}_{0}-\frac{1}{\sqrt2}\right) = -\frac{1}{\sqrt2}$. $\boxed{\text{Series C's to } -\tfrac{1}{\sqrt2}}$

[5] (10) $\sum_{k=1}^{\infty}\left(\frac{\ln k}{k^2+13}\right)^k$ ROOT TEST (positive series)

$\rho = \lim_{k\to\infty}\left(\left(\frac{\ln k}{k^2+13}\right)^k\right)^{1/k} = \lim_{k\to\infty}\frac{\ln k\to\infty}{k^2+13\to\infty} = \lim_{x\to\infty}\frac{\ln x\to\infty}{x^2+13\to\infty}$

$\overset{L}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0 < 1$

$\therefore$ $\boxed{\text{Series C's by the ROOT TEST.}}$

6 (10) $\sum_{k=1}^{\infty} \frac{20k^3 + 200k^2}{k^5+1} \sim \sum_{k=1}^{\infty} \frac{k^3}{k^5} = \sum_{k=1}^{\infty} \frac{1}{k^2}$

positive series

LCT $L = \lim_{k\to\infty} \frac{20k^3+200k^2}{k^5+1} \cdot \frac{k^2}{1} = \lim_{k\to\infty} \frac{20k^5+200k^4}{k^5+1} = \lim_{k\to\infty} \frac{20 + \overset{\to 0}{200/k}}{1+\underset{\to 0}{1/k^5}} = 20$

Since $0 < 20 < +\infty$ and since $\sum \frac{1}{k^2}$ is a convergent p $\sum$ $(p=2>1)$, $\boxed{\text{series C's by LCT}}$

7 (10) $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{5n!}$ I. AbC? RATFACE $\sum_{n=1}^{\infty} \frac{3^n}{5n!}$

$r = \lim_{n\to\infty} \frac{3^{n+1}}{5(n+1)!} \cdot \frac{5n!}{3^n} = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{5n!}{5(n+1)n!} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$ $\boxed{\text{Series is } \underline{\text{AbC}}}$

8 (10) $\sum_{n=1}^{\infty} (-1)^n \frac{6n^2+10}{n+10}$ Divergence Test: $\lim_{n\to\infty} \frac{(-1)^n 6n^2+10}{n+10}$

(i) $\lim_{n\to\infty} \frac{6n^2+10}{n+10} = \lim_{n\to\infty} \frac{6n + \overset{\to 0}{10/n}}{1+\underset{\to 0}{10/n}} \to \infty$

(ii) $\lim_{n\to\infty} \frac{(-1)^n 6n^2+10}{n+10}$ does not exist - it oscillates: odd terms $\to -\infty$, even terms $\to +\infty$ $\therefore$ $\boxed{\text{Series D's by DV. Test}}$

9 (6) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{10n^2}$ $|S - S_n| \le b_{n+1} < \frac{1}{100}$

$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$

Solve for n: $\frac{1}{10(n+1)^2} < \frac{1}{100}$

$10(n+1)^2 > 100$

$(n+1)^2 > 10$

$n+1 > \sqrt{10}$

$n > \sqrt{10} - 1$ so $n > 2$ so $\underline{\underline{n=3}}$

Now find $S_3 = \frac{1}{10} - \frac{1}{40} + \frac{1}{90}$

$= \frac{36 - 9 + 4}{360} = \frac{31}{360}$ $\therefore$ $\boxed{S \approx S_3 = \frac{31}{360}}$