

1. (24 points) Each of the following parts is worth 6 points.

(a) Use a u-substitution to help you find  $\int \frac{x}{\sqrt{x+2}} dx$

$$u = x+2 \quad du = dx \quad x = u-2$$

$$\int \frac{x}{\sqrt{x+2}} dx = \int \frac{u-2}{\sqrt{u}} du$$

$$= \int u du - 2 \int \frac{1}{\sqrt{u}} du$$

$$= \frac{2}{3} u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C$$

$$\boxed{= \frac{2}{3} (x+2)^{\frac{3}{2}} - 4(x+2)^{\frac{1}{2}} + C}$$

$$= \frac{2}{3} (x-4) \sqrt{x+2} + C$$

(b) Multiply by 1 to help you find  $\int \frac{1}{1+\cos x} dx$

$$\int \frac{1}{1+\cos x} dx = \int \frac{1}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} dx$$

$$= \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx - \int \frac{\cos x}{\sin^2 x} dx = \int \csc^2 x dx - \int \frac{\cos x}{\sin^2 x} dx$$

$$= -\cot x - \int \frac{du}{u^2} \quad (u = \sin x, \ du = \cos x dx)$$

$$= -\cot x + u^{-1} + C$$

$$\boxed{= -\cot x + \frac{1}{\sin x} + C} \quad \boxed{= -\cot x + \csc x + C}$$

$$= \frac{1-\cos x}{\sin x} + C = \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} + C$$

$$= \tan \left( \frac{x}{2} \right) + C$$

(c) Complete a square to help you find  $\int \frac{1}{x^2 + 2x + 5} dx$ . DO NOT SIMPLIFY YOUR ANSWER.

$$\begin{aligned}\int \frac{1}{x^2 + 2x + 5} dx &= \int \frac{1}{(x+1)^2 + 2^2} dx \\&= \int \frac{1}{u^2 + 2^2} du \quad (u = x+1, \ du = dx) \\&= \frac{1}{2} \tan^{-1} \frac{u}{2} + C \\&= \frac{1}{2} \tan^{-1} \left( \frac{x+1}{2} \right) + C\end{aligned}$$

(d) Short answers.

i) State an identity that allows you to convert the integrand of  $\int \sin x \sqrt{\frac{2}{1 + \cos 2x}} dx$  to  $\tan x$ .

$$\frac{1 + \cos 2x}{2} = \cos^2 x$$

ii)  $\int \tan x dx = -\ln |\cos x| + C$

Evaluate the following integrals. **Do not simplify your answers.**

2. (10 points)  $\int_0^1 x(1-x^2)^{5/2} dx$

$$u = 1 - x^2, \quad du = -2x dx,$$

$$x=0 \Rightarrow u=1-0=1$$

$$x=1 \Rightarrow u=1-1=0$$

$$\int_0^1 x(1-x^2)^{5/2} dx = -\frac{1}{2} \int_1^0 u^{5/2} du$$

$$= \frac{1}{2} \int_0^1 u^{5/2} du = \left( \frac{1}{2} \cdot \frac{2}{7} u^{\frac{7}{2}} \right) \Big|_0^1$$

$$= \frac{1}{7} (u^{\frac{7}{2}}) \Big|_0^1 = \frac{1}{7} (1^{\frac{7}{2}} - 0^{\frac{7}{2}})$$

$$= \frac{1}{7}$$

$$\begin{aligned} x &= \sin \theta \quad dx = \cos \theta d\theta \\ x=1 &\Rightarrow 1=\sin \theta \Rightarrow \theta = \frac{\pi}{2} \\ x=0 &\Rightarrow 0=\sin \theta \Rightarrow \theta = 0 \\ &= \int_0^{\frac{\pi}{2}} \sin \theta (1-\sin^2 \theta)^{\frac{5}{2}} \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \sin \theta \cos^6 \theta d\theta \\ &= - \int_0^{\frac{\pi}{2}} \cos^6 \theta d\cos \theta \\ &= -\frac{1}{7} \cos^7 \theta \Big|_0^{\frac{\pi}{2}} \\ &= \frac{1}{7} \end{aligned}$$

3. (10 points)  $\int \frac{x^3}{x-1} dx$

$$\int \frac{x^3}{x-1} dx = \int \left( x^2 + x + 1 + \frac{1}{x-1} \right) dx \quad \begin{array}{l} x-1 \sqrt{x^3+0+0+0} \\ \rightarrow \underline{x^3-x^2} \end{array}$$

$$= \int x^2 dx + \int x dx + \int 1 dx + \int \frac{dx}{x-1} \quad \begin{array}{l} x^2+0+0 \\ \rightarrow \underline{x^2-x+0} \\ x+0 \end{array}$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x-1| + C \quad \begin{array}{l} \rightarrow x-1 \\ 1 \end{array}$$

4. (10 points)  $\int y^2 \sin^{-1} y dy$  (DO NOT SIMPLIFY YOUR ANSWER)

$$\begin{aligned} u &= \sin^{-1} y & v &= \frac{1}{3} y^3 \\ du &= \frac{1}{\sqrt{1-y^2}} dy & dv &= y^2 dy \end{aligned}$$

$$= \int u dv = uv - \int v du$$

$$= \frac{1}{3} y^3 \sin^{-1} y - \frac{1}{3} \int \frac{y^3}{\sqrt{1-y^2}} dy$$

$$= \frac{1}{3} y^3 \sin^{-1} y - \frac{1}{3} \int \frac{\sin^3 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta \quad \begin{pmatrix} y = \sin \theta \\ dy = \cos \theta d\theta \end{pmatrix}$$

$$= \frac{1}{3} y^3 \sin^{-1} y - \frac{1}{3} \int \sin^3 \theta d\theta$$

$$= \frac{1}{3} y^3 \sin^{-1} y - \frac{1}{3} \int (1 - \cos^2 \theta) \sin \theta d\theta$$

$$= \frac{1}{3} y^3 \sin^{-1} y + \frac{1}{3} \int (1 - w^2) dw \quad \begin{pmatrix} w = \cos \theta \\ dw = -\sin \theta d\theta \end{pmatrix}$$

$$= \frac{1}{3} y^3 \sin^{-1} y + \frac{1}{3} \left( w - \frac{1}{3} w^3 \right) + C$$

$$= \frac{1}{3} y^3 \sin^{-1} y + \frac{1}{3} \cos \theta - \frac{1}{9} \cos^3 \theta + C$$

$$= \frac{1}{3} y^3 \sin^{-1} y + \frac{1}{3} \sqrt{1-y^2} - \frac{1}{9} (1-y^2)^{\frac{3}{2}} + C$$

$$\begin{array}{c} \text{Diagram of a right triangle with hypotenuse 1, angle } \theta \text{ at the bottom left, and vertical leg } y. \text{ The horizontal leg is labeled } \sqrt{1-y^2}. \end{array}$$

$$\begin{aligned} \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{\sqrt{1-y^2}}{1} \\ &= \sqrt{1-y^2} \end{aligned}$$

5. (10 points)  $\int \frac{1}{\sqrt{1+9x^2}} dx$

$$\int \frac{1}{\sqrt{1+9x^2}} dx$$

$$x = \frac{1}{3} \tan \theta$$

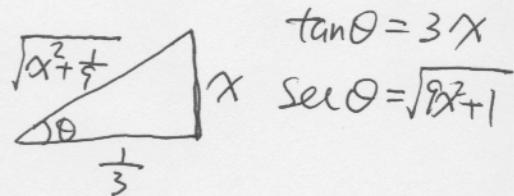
$$dx = \frac{1}{3} \sec^2 \theta d\theta$$

$$= \int \frac{1}{\sqrt{1+9 \cdot \frac{1}{9} \tan^2 \theta}} \cdot \frac{1}{3} \sec^2 \theta d\theta$$

$$= \frac{1}{3} \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \frac{1}{3} \int \sec \theta d\theta$$

$$= \frac{1}{3} \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{1}{3} \ln |\sqrt{9x^2+1} + 3x| + C$$



6. (10 points) Use partial fractions to evaluate  $\int \frac{4}{x^4+2x^2} dx$ . (DO NOT SIMPLIFY YOUR ANSWER)

$$x^4 + 2x^2 = x^2(x^2 + 2)$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2} = \frac{4}{x^4+2x^2}$$

$$\Rightarrow Ax(x^2+2) + B(x^2+2) + (Cx+D)x^2 = 4$$

$$\Rightarrow (A+C)x^3 + (B+D)x^2 + 2Ax + 2B = 4$$

$$\Rightarrow \begin{cases} A+C=0 \\ B+D=0 \\ 2A=0 \\ 2B=4 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=2 \\ C=0 \\ D=-2 \end{cases}$$

$$\int \frac{4}{x^4+2x^2} dx = \int \left( \frac{2}{x^2} - \frac{2}{x^2+2} \right) dx$$

$$= 2 \int \frac{dx}{x^2} - 2 \int \frac{dx}{x^2+2}$$

$$= -\frac{2}{x} - \sqrt{2} \tan^{-1} \left( \frac{x}{\sqrt{2}} \right) + C$$

7. (10 points) Solve for  $\theta$  in the following initial value problem. Show all your work.

$$\frac{d\theta}{dt} = \sec \theta, \theta \in (-\pi/2, \pi/2) \quad \text{and} \quad \theta(0) = \pi/3.$$

$$\int \frac{1}{\sec \theta} d\theta = \int dt$$

$$\Rightarrow \int \cos \theta d\theta = \int dt$$

$$\Rightarrow \sin \theta = t + C$$

$$\Rightarrow \theta(t) = \sin^{-1}(t + C)$$

$$\theta(0) = \frac{\pi}{3} \Rightarrow \sin^{-1}(0 + C) = \frac{\pi}{3}$$

$$\Rightarrow C = \frac{\sqrt{3}}{2}$$

Then  $\theta(t) = \sin^{-1}\left(t + \frac{\sqrt{3}}{2}\right)$

8. (16 points) Evaluate each of the following improper integrals or show that it diverges.

$$(a) \int_0^{\pi/2} \tan x \sec^3 x \, dx$$

$$= \lim_{C \rightarrow \frac{\pi}{2}^-} \int_0^C \tan x \sec^3 x \, dx$$

$$u = \sec x$$

$$= \lim_{C \rightarrow \frac{\pi}{2}^-} \int_1^{\sec(C)} u^2 \, du$$

$$du = \sec x \tan x \, dx$$

$$x=0 \Rightarrow u=\sec(0)=1$$

$$x=C \Rightarrow u=\sec(C)$$

$$= \lim_{C \rightarrow \frac{\pi}{2}^-} \left. \frac{1}{3} u^3 \right|_1^{\sec(C)}$$

$$= \lim_{C \rightarrow \frac{\pi}{2}^-} \left. \frac{1}{3} \sec^3 x \right|_0^C = \lim_{C \rightarrow \frac{\pi}{2}^-} \left( \frac{1}{3} \sec^3(C) - \frac{1}{3} \right)$$

$$= \infty$$

Therefore, the improper integral diverges.

$$(b) \int_1^\infty x^4 e^{-x^5} \, dx = \lim_{d \rightarrow \infty} \int_1^d x^4 e^{-x^5} \, dx$$

$$\text{Evaluate: } \int x^4 e^{-x^5} \, dx \quad (u = x^5, \, du = 5x^4 \, dx)$$

$$= \frac{1}{5} \int e^{-u} \, du = -\frac{1}{5} e^{-u} + C = -\frac{1}{5} e^{-x^5} + C$$

Therefore:

$$\int_1^\infty x^4 e^{-x^5} \, dx = \lim_{d \rightarrow \infty} \int_1^d x^4 e^{-x^5} \, dx = \lim_{d \rightarrow \infty} \left( -\frac{1}{5} e^{-x^5} \right) \Big|_1^d$$

$$= \lim_{d \rightarrow \infty} \left( -\frac{1}{5} e^{-d^5} - \left( -\frac{1}{5} e^{-1^5} \right) \right) = \lim_{d \rightarrow \infty} \left( -\frac{1}{5} e^{-d^5} \right) + \frac{1}{5} e^{-1}$$

$$= \frac{1}{5e}$$