KEY

Math 34 **Calculus II**

TUFTS UNIVERSITY **Department of Mathematics**

February 20, 2014 **Exam I**

No books, notes, **or calculators**. **TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You **are not allowed to** use reduction formulas. You may not use the tabular method for integration by parts problems. Even when told not to simplify, you are not allowed to leave answers in unsimplified forms like $\cos^{-1}(\sin(2x))$. You are not allowed to quote results about growth rates. You are required to **sign** your exam book. With your signature, you pledge that you have neither given nor received assistance on this exam.

Problem	Point Value	Points
1	12	
2	10	
3	10	
4	10	
5	10	
6	10	
7	16	
8	10	
9	12	
	100	

1. (12 points) Let S be the region bounded by the curves $y = x^2$ and $y = x + 2$.

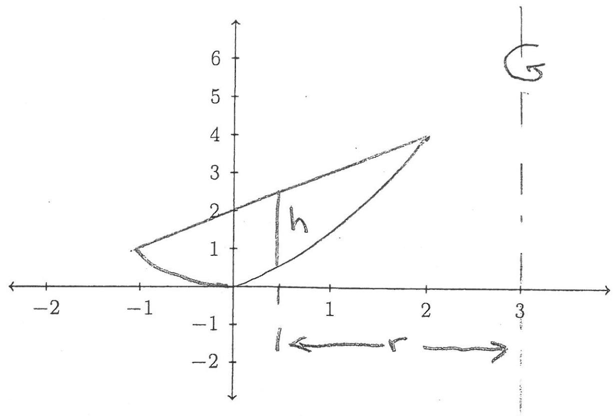
(a) Sketch the region S and the vertical line $x = 3$. Set up but *do not evaluate* an expression that gives the volume of the solid obtained by rotating the region S about the vertical line $x = 3$.

$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

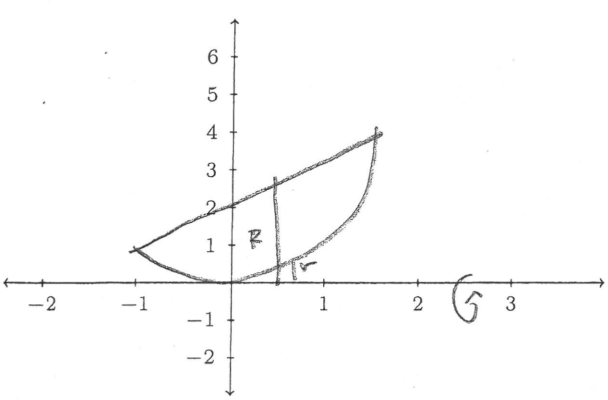
$x = 2, -1$



Shells

$$V = 2\pi \int_{-1}^{2} r h \, dx = 2\pi \int_{-1}^{2} (3-x)\left[(x+2) - x^2\right] dx$$

(b) Sketch the same region S. Set up but *do not evaluate* an expression that gives the volume of the solid obtained by rotating the region S about the x-axis.



Washer

$$V = \pi \int_{-1}^{2} (R^2 - r^2) \, dx$$

$$= \pi \int_{-1}^{2} \left[(x+2)^2 - (x^2)^2\right] dx$$

Evaluate the following integrals. **Do not simplify your answers.**

2. (10 points) $\int \tan^3(x)\sec^4(x)\,dx$ $= \int \tan^3 x \sec^2 x \sec^2 x\,dx$

$= \int \tan^3 x(\tan^2 x+1)\sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$= \int u^3(u^2+1)\,du$

$= \int (u^5+u^3)\,du$

$= u^6/6 + u^4/4 + C = \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C$

3. (10 points) $\int t^2 \sin(t)\,dt$ $= -t^2\cos t + 2\int t\cos t\,dt$

PARTS $u = t^2$ $dv = \sin t\,dt$

$du = 2t\,dt$ $v = -\cos t$

Parts

$u = t$ $dv = \cos t\,dt$

$du = dt$ $v = \sin t$

$= -t^2\cos t + 2\left[t\sin t - \int \sin t\,dt\right]$

$= -t^2\cos t + 2\left[t\sin t - -\cos t\right] + C$

$= -t^2\cos t + 2t\sin t + 2\cos t + C$

4. (10 points) $\displaystyle\int_0^{\sqrt{2}} \frac{x^2\,dx}{\sqrt{4-x^2}}$

$x = 2\sin\theta \quad -\pi/2 \le \theta \le \pi/2$

$dx = 2\cos\theta\,d\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = |2\cos\theta| = 2\cos\theta$

<u>bnds</u> $\sin\theta = \frac{x}{2}$

$x = \sqrt{2} \quad \sin\theta = \frac{\sqrt{2}}{2}, \; \theta = \pi/4$

$x = 0 \quad \sin\theta = 0, \; \theta = 0$

$$= \int_0^{\pi/4} \frac{4\sin^2\theta \cdot 2\cos\theta\,d\theta}{2\cos\theta} = \int_0^{\pi/4} 4\sin^2\theta\,d\theta = \frac{4}{2}\int_0^{\pi/4} (1-\cos 2\theta)\,d\theta$$

$$= 2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]\Big|_0^{\pi/4} = 2\left[\left(\pi/4 - \tfrac{1}{2}\sin \pi/2\right) - \left(0 - \tfrac{1}{2}\sin 0\right)\right]$$

$$= 2\left[\pi/4 - \tfrac{1}{2}\right] = \boxed{\frac{\pi}{2} - 1}$$

5. (10 points) $\displaystyle\int \frac{dx}{(x^2+9)^{3/2}}$ $\displaystyle = \int \frac{3\sec^2\theta\,d\theta}{27\sec^3\theta} = \frac{1}{9}\int \frac{d\theta}{\sec\theta} = \frac{1}{9}\int \cos\theta\,d\theta$

$x = 3\tan\theta \quad -\pi/2 < \theta < \pi/2$

$dx = 3\sec^2\theta\,d\theta$

$(x^2+9)^{3/2} = (9\tan^2\theta + 9)^{3/2}$

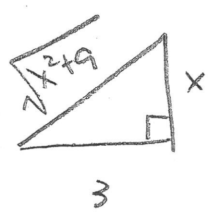
$= (9\sec^2\theta)^{3/2}$

$= 9^{3/2}(\sec^2\theta)^{3/2}$

$= 27\sec^3\theta$

$= \frac{1}{9}\sin\theta + C$

$\tan\theta = x/3$



$$= \frac{1}{9}\left(\frac{x}{\sqrt{x^2+9}}\right) + C$$

6. (10 points) $\int \frac{x^2+2x+2}{(x-1)(x^2+4)}\,dx$

PF's $\quad \frac{x^2+2x+2}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

$(*)\quad x^2+2x+2 = A(x^2+4) + (Bx+C)(x-1)$

$x=1 \qquad 1+2+2 = A(1+4)$

$\qquad 5 = 5A \rightarrow A=1$

$x^2: \qquad 1 = A+B \rightarrow B=0$

Constant: $\qquad 2 = 4A - C$

$\qquad 2 = 4 - C \rightarrow C=2$

$$\int \frac{dx}{x-1} + \int \frac{2\,dx}{x^2+4}$$

$$= \ln|x-1| + \frac{2}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \ln|x-1| + \tan^{-1}\frac{x}{2} + C$$

7. (16 points) Evaluate the following integrals or state that they diverge.

(a) $\displaystyle\int_0^8 \frac{dx}{\sqrt[3]{8-x}}$ $= \displaystyle\lim_{b\to 8^-} \int_0^b (8-x)^{-1/3}\,dx = -\frac{3}{2}\lim_{b\to 8^-}\left[(8-x)^{2/3}\right]\Big|_0^b$

$[0,8)$

$\int (8-x)^{-1/3}\,dx$

$u=8-x$

$du=-dx$

$-du=dx$

$-\int u^{-1/3}\,du = -\frac{3}{2}u^{2/3}+c$

$= -\frac{3}{2}(8-x)^{2/3}+c$

$= -\frac{3}{2}\displaystyle\lim_{b\to 8^-}\left[\underbrace{(8-b)^{2/3}}_{\to 0} - (8-0)^{2/3}\right]$

$= -\frac{3}{2}\left[0 - 8^{2/3}\right]$

$= -\frac{3}{2}\left[-4\right] = \boxed{6}$

(b) $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$ $= \displaystyle\lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x}$ $= \displaystyle\lim_{t\to\infty}\left[\ln|\ln x|\right]\Big|_2^t$

$\int \frac{dx}{x\ln x}$

$u=\ln x$

$du=\frac{1}{x}dx$

$\int \frac{du}{u} = \ln|u|+c$

$= \ln|\ln x|+c$

$= \displaystyle\lim_{t\to\infty}\left[\ln|\ln t| - \ln|\ln 2|\right]$

$\downarrow$

∞

Integral diverges

8. (10 points) Solve for y in the following initial value problem. Show all work.

$$yy' = xe^{-y^2}, \quad y(0) = -2.$$

$$y\,\frac{dy}{dx} = xe^{-y^2}$$

$$ye^{y^2}\,dy = x\,dx$$

$$\int ye^{y^2}\,dy = \int x\,dx$$

$u = y^2$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

$$\frac{1}{2}\int e^u\,du = \int x\,dx$$

$$\frac{1}{2}e^u = \frac{x^2}{2} + C$$

$$\frac{1}{2}e^{y^2} = \frac{x^2}{2} + C$$

rename C

$$e^{y^2} = x^2 + C$$

$$\ln e^{y^2} = \ln(x^2 + C)$$

$$y^2 = \ln(x^2 + C)$$

$$y = \pm\sqrt{\ln(x^2 + C)}$$

$$\boxed{y = -\sqrt{\ln(x^2 + e^4)}}$$

Solve for C: $y(0) = -2$

$$-2 = -\sqrt{\ln(C)}$$

$$4 = \ln C \rightarrow e^4 = C$$

9. (12 points) Find the limits of the following sequences or state that the limit does not exist. Show all work.

(a) $a_n = \dfrac{1-2n-5n^3}{14n^2+13n^3}$

$$\lim_{n\to\infty} \frac{1-2n-5n^3}{14n^2+13n^3} = \lim_{n\to\infty} \frac{\underset{\to 0}{1/n^3} - \underset{\to 0}{2/n^2} - 5}{\underset{\to 0}{14/n} + 13} = \frac{-5}{13}$$

divide all terms by n^3

(b) $a_n = \left(1-\dfrac{1}{n}\right)^{3n}$

"Form" 1^∞

① $\lim_{n\to\infty} \ln\left(1-\frac{1}{n}\right)^{3n}$

$= \lim_{n\to\infty} 3n\ln\left(1-\frac{1}{n}\right)$

$= 3\lim_{n\to\infty} \underset{\downarrow\,\infty}{n}\ \underset{\downarrow\,0}{\ln\left(1-\frac{1}{n}\right)} = 3\lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)\to 0}{\frac{1}{n}\to 0}$

$= 3\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)\to 0}{\frac{1}{x}\to 0} \overset{L}{=} 3\lim_{x\to\infty}\frac{\left(\frac{1}{1-\frac{1}{x}}\right)\left(\frac{1}{x^2}\right)}{\left(-1/x^2\right)}$

$= 3\lim_{x\to\infty}\frac{-1}{1-\underset{\to 0}{1/x}} = -3$

② $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{3n} = \boxed{e^{-3}}$

Math 34 Exam I February 20, 2013

Name ________________________________

Circle your section:

32-01 Hao Liang TThF 8:30-9:20

32-02 Mary Glaser TThF 12-12:50

32-03 Gail Kaufmann TTHF 12-12:50

32-04 Thomas Benson TTh1:30-2:20,F2:30-3:20

I pledge that I have neither given nor received assistance on this exam.

Signature ________________________________