SOLUTIONS

Math 34
Calculus II

TUFTS UNIVERSITY
Department of Mathematics

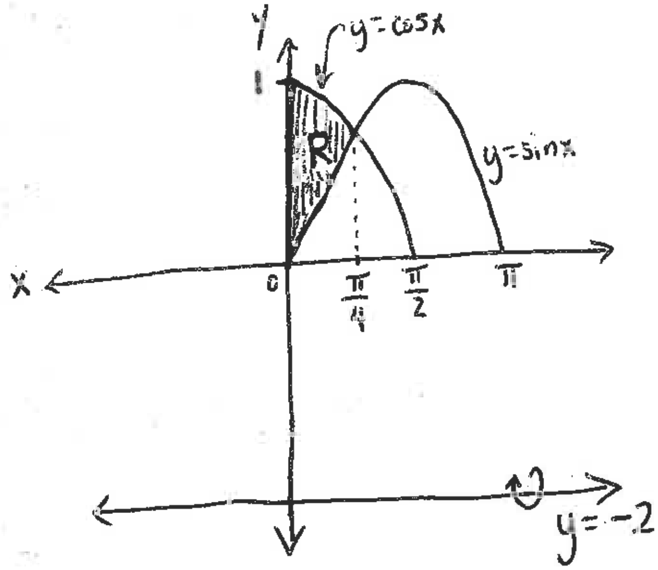
October 6, 2014
Exam I
12-1:20

No books, notes, **or calculators. TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You are **not allowed to** use reduction formulas. You may not use the tabular method for integration by parts problems. Even when told not to simplify, you are not allowed to leave answers in unsimplified forms like $\cos^{-1}(\sin(2x))$. You are not allowed to quote results about growth rates. You are required to **sign** your exam on the last page. With your signature, you pledge that you have neither given nor received assistance on this exam.

Problem	Point Value	Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	16	
8	10	
9	14	
	100	

1. (10 points) Let R be the region next to the y-axis whose three sides are bounded by the curves $y = \sin x$, $y = \cos x$, and $x = 0$.

(a) Sketch the region R and the horizontal line $y = -2$. Set up but *do not evaluate* a single integral that gives the volume of the solid obtained by rotating the region R about the horizontal line $y = -2$.



Washer

$$V = \pi \int_0^{\frac{\pi}{4}} \left[(\cos x + 2)^2 - (\sin x + 2)^2 \right] dx$$

(b) For the same region R, set up but *do not evaluate* a single integral that gives the volume of the solid obtained by rotating the region R about the y-axis.

Shell

$$V = 2\pi \int_0^{\frac{\pi}{4}} x(\cos x - \sin x)\, dx$$

Evaluate the following integrals. **Do not simplify your answers.**

2. (10 points) $\int (5 + \sin^2 3x)\, dx$ $= \int 5\, dx + \int \sin^2 3x\, dx$

$= 5x + \int \sin^2 3x\, dx$

$u = 3x$

$du = 3dx \rightarrow \frac{1}{3}du = dx$

$= 5x + \int \frac{1}{3}\sin^2 u\, du \quad = \quad 5x + \frac{1}{3}\int \left(\frac{1-\cos 2u}{2}\right) du$

$= 5x + \frac{1}{6}\int (1-\cos 2u)\, du$

$= 5x + \frac{1}{6}\left(u - \frac{1}{2}\sin 2u\right) + C$

$= 5x + \frac{u}{6} - \frac{\sin 2u}{12} + C$

Final Answer: $= \boxed{5x + \frac{x}{2} - \frac{\sin 6x}{12} + C}$

3. (10 points) $\int x \sec(x) \tan(x)\, dx$

Integration by Parts:

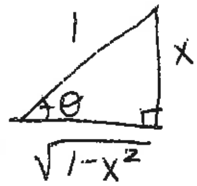
$u = x \qquad dv = \sec(x)\tan(x)dx$

$du = dx \qquad v = \sec(x)$

$\int x\sec(x)\tan(x)dx = x\sec(x) - \int \sec(x)dx$

$= \boxed{x\sec(x) - \ln|\sec(x) + \tan(x)| + C}$

4. (10 points) $\int_0^1 x^3\sqrt{1-x^2}\,dx$



$x = \sin\theta$
$dx = \cos\theta\, d\theta$

Change bounds
$x = 0 = \sin\theta \rightarrow \theta = 0$
$x = 1 = \sin\theta \rightarrow \theta = \frac{\pi}{2}$

$$\int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sin^3\theta\sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\sin\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\cos^2\theta\sin\theta\, d\theta$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

Change bounds
$\cos(0) = 1$
$\cos(\frac{\pi}{2}) = 0$

$$= -\int_1^0 (1-u^2)u^2\, du$$

$$= -\int_1^0 (u^2-u^4)\, du$$

$$= -\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_1^0$$

$$= -\left[(0) - \left(\frac{1}{3} - \frac{1}{5}\right)\right] = -\left[-\left(\frac{5}{15} - \frac{3}{15}\right)\right]$$

$$= \boxed{2/15}$$

5. (10 points) $\int e^x \tan^3(e^x)\,dx = \int \tan^3(u)\,du$

$u = e^x$

$du = e^x dx$

$$= \int \tan u(\sec^2 u - 1)\,du$$

$$= \underbrace{\int \tan u \sec^2 u\,du}_{\substack{w = \tan u \\ dw = \sec^2 u\,du}} - \int \tan u\,du$$

$$= \int w\,dw - \ln|\sec u| + C$$

$$= \frac{w^2}{2} - \ln|\sec u| + C$$

$$= \frac{\tan^2 u}{2} - \ln|\sec u| + C$$

$$= \boxed{\frac{\tan^2(e^x)}{2} - \ln|\sec(e^x)| + C}$$

6. (10 points) $\int \frac{6}{(x+2)(x^2+2)}\,dx$

Partial Fractions

$$\left(\frac{6}{(x+2)(x^2+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2}\right)(x+2)(x^2+2)$$

$$6 = A(x^2+2) + (Bx+c)(x+2)$$

$$6 = (A+B)x^2 + (2B+C)x + 2A + 2C$$

Plug in $x=-2$: $6 = 6A$

$\boxed{A=1}$

Look at constants:

$6 = 2A + 2C$

$6 = 2(1) + 2C$

$4 = 2C$

$\boxed{C=2}$

Look at x^2

$0 = A + B$

$0 = 1 + B$

$\boxed{B=-1}$

$$\int \frac{6}{(x+2)(x^2+2)}dx = \int \frac{1}{x+2}dx + \int \frac{2-x}{x^2+2}dx$$

$$= \int \frac{1}{x+2}dx + 2\int \frac{1}{x^2+2}dx - \underbrace{\int \frac{x}{x^2+2}dx}$$

$u = x^2+2$

$\frac{1}{2}du = x\,dx$

$$= \int \frac{1}{x+2}dx + 2\int \frac{1}{(x^2+(\sqrt{2})^2)} - \frac{1}{2}\int \frac{1}{u}du$$

$$= \ln|x+2| + 2\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right) - \frac{1}{2}\ln|u| + C$$

$$= \boxed{\ln|x+2| + \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2}\ln|x^2+2| + C}$$

7. (16 points) Evaluate each of the following improper integrals or show that it diverges.

(a) $\int_0^\infty \frac{(\tan^{-1} x)^2}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{(\tan^{-1}x)^2}{1+x^2}\,dx$

$u = \tan^{-1}x$

$du = \frac{dx}{1+x^2}$

Change bounds:

$x=0: \tan^{-1}(0)=0$

$x=b: \tan^{-1}b$

$= \lim_{b\to\infty} \int_0^{\tan^{-1}b} u^2\,du$

$= \lim_{b\to\infty} \frac{u^3}{3}\Big|_0^{\tan^{-1}b}$

$= \lim_{b\to\infty} \left(\frac{(\tan^{-1}b)^3}{3} - 0\right) = \frac{\left(\frac{\pi}{2}\right)^3}{3} = \boxed{\frac{\pi^3}{24}}$

(b) $\int_1^{e^7} \frac{1}{x\ln x}\,dx = \lim_{c\to1^+} \int_c^{e^7} \frac{1}{x\ln x}\,dx$

$u = \ln x$

$du = \frac{dx}{x}$

Change bounds:

$x=e^7: \ln(e^7) = 7$

$x=c: \ln(c)$

$= \lim_{c\to1^+} \int_{\ln(c)}^{7} \frac{1}{u}\,du$

$= \lim_{c\to1^+} \ln|u|\Big|_{\ln(c)}^{7}$

$= \lim_{c\to1^+} \left(\ln 7 - \ln|\ln(c)|\right)$

As $c\to1^+$, $\ln(c)\to 0$, so $\ln|\ln(c)| \to -\infty$

$= +\infty$ DIVERGES

8. (10 points) Solve for y in the following initial value problem. Show all work.

$$y' = e^{-t}\sqrt{1-y^2}\ ,\ t \geq 0 \qquad \text{and} \qquad y(0) = 1/2.$$

$$\frac{dy}{dt} = e^{-t}\sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = e^{-t}dt$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int e^{-t}dt$$

$$\sin^{-1} y = -e^{-t} + C$$

$$\underline{y = \sin(-e^{-t} + c)}$$

$$y(0) = \frac{1}{2} = \sin(-e^{0} + c)$$

$$\frac{1}{2} = \sin(-1 + c)$$

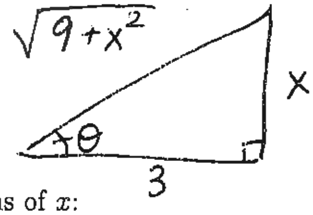
$$\Rightarrow c - 1 = \frac{\pi}{6}$$

$$+1 \qquad +1$$

$$\underline{c = \frac{\pi}{6} + 1}$$

$$\therefore \boxed{y(t) = \sin\left(-e^{-t} + \frac{\pi}{6} + 1\right)}$$

$\tan\theta = \frac{x}{3}$



9. (14 points)

(a) (6 points) If $x = 3\tan\theta$, express each of the following in terms of x:

i) $\theta =$

$x = 3\tan\theta$

$\frac{x}{3} = \tan\theta$

$\boxed{\theta = \tan^{-1}\left(\frac{x}{3}\right)}$

ii) $\sec\theta = \frac{\text{hyp.}}{\text{adj.}}$

$= \boxed{\frac{\sqrt{9+x^2}}{3}}$

iii) $\sin(2\theta) = 2\sin\theta\cos\theta$

$\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$= 2\left(\frac{x}{\sqrt{9+x^2}}\right)\left(\frac{3}{\sqrt{9+x^2}}\right) = \boxed{\frac{6x}{9+x^2}}$

(b) (4 points) Find the form of the partial fractions decomposition of the rational function $\frac{x^7+1}{x^5+4x^3}$, but DO NOT SOLVE FOR THE CONSTANTS.

$$\begin{array}{r} x^2 - 4 \\ x^5+4x^3 \enclose{longdiv}{x^7 + 1} \\ -(x^7+4x^5) \\ \hline -4x^5+1 \\ -(-4x^5-16x^3) \\ \hline 16x^3+1 \end{array}$$

$$\frac{x^7+1}{x^5+4x^3} = x^2 - 4 + \frac{16x^3+1}{x^5+4x^3}$$

$$= x^2 - 4 + \frac{16x^3+1}{x^3(x^2+4)}$$

$$= \boxed{(x^2-4) + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}}$$

(c) (4points)

i) Does $b_n = \frac{2n + (-1)^n n}{2n}$ converge or diverge? Why?

$$= \frac{2n}{2n} + \frac{(-1)^n n}{2n} = 1 + \frac{(-1)^n}{2}$$

Terms alternate between $\frac{1}{2}$ and $\frac{3}{2}$, so DIVERGES.

ii) Find a recurrence relation that generates the sequence: $\{1/1, 1/2, 1/6, 1/24, 1/120, 1/720, \ldots\}$

$a_1 = \frac{1}{1}$, $a_{n+1} = \frac{a_n}{n+1}$ for $n = 1, 2, 3, \ldots$

Since $\frac{a_2}{a_1} = \frac{1/2}{1/1} = \frac{1}{2}$, $\frac{a_3}{a_2} = \frac{1/6}{1/2} = \frac{1}{3}$, $\frac{a_4}{a_3} = \frac{1/24}{1/6} = \frac{1}{4}$

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Name ______________________________

Circle your section:

34-01 Molly Hahn TThF 8:30-9:20

34-02 Melody Takeuchi

34-03 Mary Glaser TThF 12-12:50

32-04 Molly Hahn TTh1:30-2:45

I pledge that I have neither given nor received assistance on this exam.

Signature ______________________________