Math 12 | Tufts University
Department of Mathematics
Final Exam | December 16, 2011
8:30-10:30am

No books, notes or calculators are allowed. Cross out what you do not want us to grade. You must show all your work in order to received full credit. Please write neatly. You are required to sign your exam book. With your signature, you pledge that you have neither given or received assistance on this exam.

1. (18 points) Integrate or evaluate. Simplify your answer in part (b).

   (a) $\int \sin^2(x/2)\,dx$ (b) $\int_0^1 \frac{x+7}{x^2-x-2}\,dx$ (c) $\int \sqrt{x}\ln x\,dx$

2. (18 points) Determine whether each of the following series converges or diverges. Justify your answer. State and check hypotheses of any test, rules or theorems you use. You may *not* simply quote a theorem.

   (a) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ (b) $\sum_{n=1}^{\infty} \frac{3n^2}{8^n}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3n^3+10}$

3. (8 points) Find the radius of convergence and interval of convergence for the following power series:
$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{4^n\sqrt{n}}$$

4. (4 points) Sketch each of the following sets of points in the polar plane. (Use separate sketches for each set.)
   (a) $\{(r,\theta) : 1 \le r < 2 \text{ and } \frac{-\pi}{4} \le \theta \le \frac{\pi}{4}\}$
   (b) $\{(r,\theta) : r \le 0 \text{ and } \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$

5. (8 points) Compute the Taylor series for $f(x) = xe^x$ centered at $a = 1$ using the definition of the Taylor series. Write the series using summation notation. Find the interval of convergence.

6. (6 points) Consider the following parametric equations:
$x = 3 - 2\cos t$ and $y = 4\sin^2 t$; $\pi \le t \le 2\pi$
   (a) Eliminate the parameter t to obtain an equation in x and y.
   (b) Sketch the curve and indicate the positive orientation.

please turn over

You may use any of the following for problems 7 and 8.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

7. (12 points) Find series representations and **radii of convergence** for each of the following functions. Write the series using summation notation.

   (a) $f(x) = \dfrac{x^4}{6+x}$

   (b) $f(x) = ln(1-2x)$

8. (4 points) Identify each of the functions represented by the following power series.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{4k}}{(2k)!}$

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+3}}{5k!}$

9. (8 points) Polar Curves

   (a) Sketch the cardioid $r = 2 + 2sin\theta$ and the circle $r = 6sin\theta$ in the same polar graph and shade the region outside the cardioid and inside the circle.

   (b) Set up but *do not evaluate* a definite integral that expresses the area of the region described in part(a).

10. (14 points) Taylor Polynomials and Remainder

    (a) Let $f(x) = ln(x)$.

    i. Find an expression for the remainder term $R_2(x)$ in the second order Taylor polynomial for $f(x) = ln(x)$ centered at $a = 1$.

    ii. Use this remainder term to estimate the absolute error in approximating $ln(1/2)$ with $p_2(1/2)$.

    (b) Find the third Taylor polynomial $p_3(x)$ for the function $f(x) = \sin(2x)$ centered at $a = \pi/3$. Evaluate all coefficients.

End of exam. Have a great holiday.