Math 11 - Final
Solutions

1. a) $y' = \dfrac{\frac{1}{x}(x^3+2) - 3x^2 \ln x}{(x^3+2)^2}$; b) $y' = \dfrac{e^x}{1+e^{2x}}$;

c) $y' = \left(e^{\frac{1}{x}\ln x}\right)' = e^{\frac{1}{x}\ln x}\left(\dfrac{1}{x^2} - \dfrac{\ln x}{x^2}\right) = \dfrac{x^{\frac{1}{x}}(1-\ln x)}{x^2}$

d) $1 \cdot y^2 + 2xy\dfrac{dy}{dx} = e^y \dfrac{dy}{dx} \Leftrightarrow \dfrac{dy}{dx}\left(e^y - 2xy\right) = y^2$

$$\Rightarrow y' = \frac{y^2}{e^y - 2xy}$$

2. a) $\lim\limits_{x\to 1} \dfrac{(x-1)(x+1)}{(x-1)(x-2)} = \lim\limits_{x\to 1} \dfrac{x+1}{x-2} = \underline{\underline{-2}}$

b) $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x^2}} \overset{L'H}{\underset{\frac{-\infty}{\infty}}{=}} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim\limits_{x\to 0^+} -\dfrac{x^2}{2} = \underline{\underline{0}}$

c) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x - \tan x} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim\limits_{x\to 0} \dfrac{1-\cos x}{1-(1+\tan^2 x)} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{-\tan^2 x} =$

$\overset{L'H}{\underset{\frac{0}{0}}{=}} \lim\limits_{x\to 0} \dfrac{\sin x}{-2\tan x(1+\tan^2 x)} = \lim\limits_{x\to 0} \dfrac{1}{\frac{-2(1+\tan^2 x)}{\cos x}} = \underline{\underline{-\dfrac{1}{2}}}$

3. a) $\lim\limits_{x\to\pm\infty} f(x) = 0 \Rightarrow \boxed{y=0}$ H.A.

b) $\lim\limits_{x\to 0} f(x) = -\infty \Rightarrow \boxed{x=0}$ V.A.

c)

x	$-\infty$	0		2	∞
$2-x$	$+ + + + +$		$+$	0	$- - -$
x^3	$- - - - - -$		$+$		$+ + +$
$f'(x)$	$-$		$+$		$-$

$f \nearrow$ on $(0, 2)$

d) $x = 2$ yields a local max (from table above)

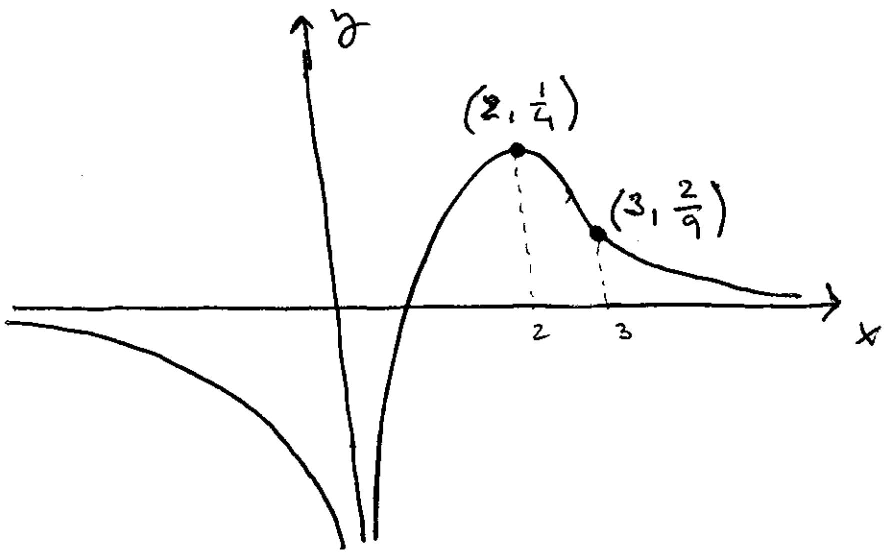
e)

x	$-\infty$	0		3	∞
$x-3$	$- - -$	/	$- -$	0	$+ + +$
f''	$- -$	/	$- - -$		$+ + +$

f concave up on $(3, \infty)$

f) $x = 3$ yields an inflection point (from table above)

g)



4a) $a(t) = \cos t - e^t - 1$, $s(0)=0$, $v(0)=5$

$$v(t) = \sin t - e^t - t + C_1$$
$$v(0) = 0 - 1 - 0 + C_1 = 5 \Rightarrow C_1 = 6$$
$$\Rightarrow v(t) = \sin t - e^t - t + 6$$

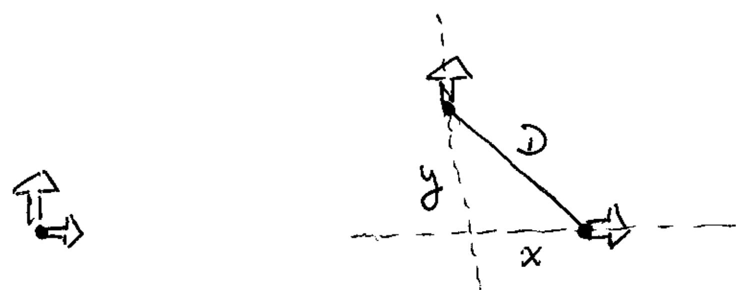
$$s(t) = -\cos t - e^t - \tfrac{1}{2}t^2 + 6t + C_2$$
$$s(0) = -1 - 1 - 0 + 0 + C_2 = 0 \Rightarrow C_2 = 2$$

So: $s(t) = -\cos t - e^t - \frac{1}{2}t^2 + 6t + 2$

b) There exists $c \in (1, 10)$ such that:

$$f'(c) = \frac{f(10) - f(1)}{10 - 1} = \frac{6+3}{9} = 1.$$

5. a)



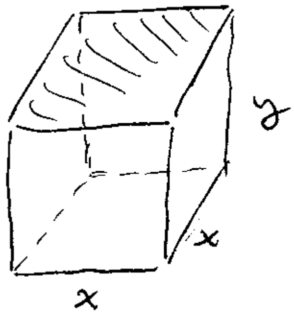
$\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 12$

$$D^2 = x^2 + y^2 \Rightarrow 2D \cdot \frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

After 1h: $x = 5$, $y = 12$, $D = \sqrt{25+144} = \sqrt{169} = 13$

So: $\frac{dD}{dt} = \frac{2 \cdot 5 \cdot 5 + 2 \cdot 12 \cdot 12}{2 \cdot 13} = \frac{25 + 144}{13} = \frac{169}{13} = \underline{\underline{13 \text{ mph}}}$

5 b)



$$A = x^2 + 4xy = 48$$

$$V = x^2 y = x^2 \frac{48 - x^2}{4x} = \frac{48x - x^3}{4}$$

$$V' = \frac{48 - 3x^2}{4} = \frac{3}{4}\left(16 - x^2\right)$$

x		-4		0		4	
V'	$---$	0	$+$	$+$	$+$	0	$---$
V					$\nearrow$		$\searrow$

V attains its maximum value when $x = \underline{\underline{4}}$.

$$y = \frac{48 - 16}{16} = \frac{32}{16} = \underline{\underline{2}}$$

6. a) $\sin^{-1}\left(\frac{1}{2}\right) = y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \frac{1}{2} = \sin y$, so $y = \underline{\underline{\frac{\pi}{6}}}$

b) $f'(0) = \left(e^x + \frac{1}{\sqrt{1 - x^2}}\right)\Bigg|_{x=0} = 1 + 1 = 2$

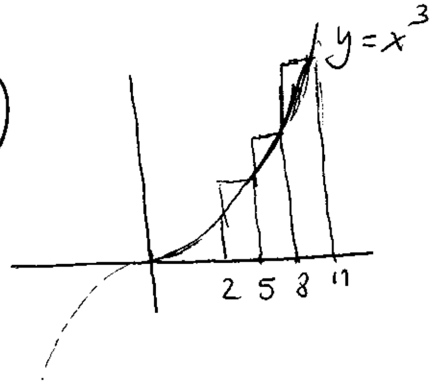
$y - 1 = 2(x - 0)$ or $\boxed{y = 2x + 1}$

c) $\arcsin x : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

<u>So</u>: $-1 \le 2x - 3 \le 1$ or $2 \le 2x \le 4$

or $1 \le x \le 2$ or $\boxed{x \in [1, 2]}$

7. a)



$$\Delta x = \frac{11-2}{3} = \frac{9}{3} = 3$$

$$3f(5) + 3f(8) + 3f(11)$$

$$= 3\left(5^3 + 8^3 + 11^3\right)$$

b) $$\int_{-3}^{3} g(x)\,dx = \int_{-3}^{-1} g(x)\,dx + \int_{-1}^{2} g(x)\,dx + \int_{2}^{3} g(x)\,dx$$

$$= \frac{2\cdot 2}{2} + \frac{1}{2}\pi(1.5)^2 - \frac{1\cdot 1}{2}$$

$$= \frac{3 + 2.25\pi}{2} = \frac{3}{2} + \frac{9}{8}\pi$$

8. a) $f'(x) = \dfrac{x}{1+x^3}$, by FTC (1).

b) $g'(x) = e^{\sin(x^2)} \cdot 2x$, by an extension of FTC (2)

c) $h'(x) = \dfrac{(x-1)(x-2)}{x-5}$

x	0		1		2	3		5	
$x-1$		− − −	0	+	+	+	+	+	+
$x-2$		− − −	−	−	0	+	+	+	+
$x-5$		− − −	−	−	−	−	−	0	+ + +
$h'(x)$		−		+		−			+
$h(x)$		↘		↗		↘			↗

local max

local maximum on $(0, 3)$ when $x = 2$

9. a) $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$ $\quad u = \frac{1}{x}$, $du = -\frac{1}{x^2}dx$ $\quad x=1 \Rightarrow u=1$, $x=2 \Rightarrow u=\frac{1}{2}$

$$= -\int_1^{\frac{1}{2}} e^u\,du = \int_{\frac{1}{2}}^1 e^u\,du = e^u\Big|_{\frac{1}{2}}^1 = e - \sqrt{e}.$$

b) $\int x\sin(x^2)\,dx \underset{\substack{u=x^2\\ du=2x\,dx}}{=} \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u) + C$

$$= -\frac{1}{2}\cos(x^2) + C$$

c) $\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \underbrace{\int \frac{x}{1+x^2}\,dx}_{\substack{u=1+x^2\\ du=2x\,dx}}$

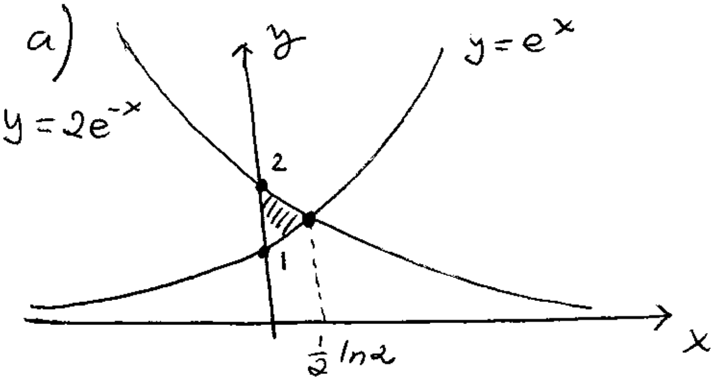
$$= \arctan(x) + \frac{1}{2}\ln(1+x^2) + C$$

d) $\int_{-1}^1 x\sqrt{x+1}\,dx$ $\quad u = x+1 \Rightarrow x = u-1$, $du = dx$ $\quad x=-1 \Rightarrow u=0$, $x=1 \Rightarrow u=2$

$$= \int_0^2 (u-1)\sqrt{u}\,du = \int_0^2 \left(u^{3/2} - u^{1/2}\right)du$$

$$= \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_0^2 = \frac{2}{5}2^{5/2} - \frac{2}{3}2^{3/2}.$$

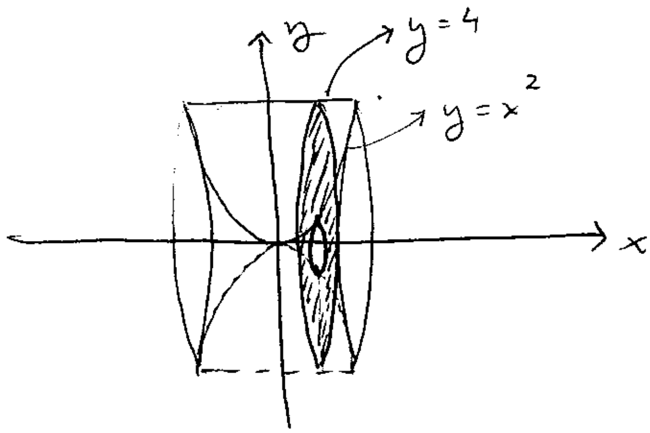
10. a)



$$e^x = 2e^{-x}$$
$$e^{2x} = 2$$
$$2x = \ln 2$$
$$x = \frac{1}{2}\ln 2$$

$$\text{Area} = \int_0^{\frac{1}{2}\ln 2} (2e^{-x} - e^x)\,dx.$$

b)



$$V = \int_{-2}^{2} \left[\pi(4^2) - \pi(x^2)^2\right] dx$$
$$= \pi \int_{-2}^{2} (16 - x^4)\,dx$$
$$= 2\pi \int_0^2 (16 - x^4)\,dx$$

by the method of washers!