KEY

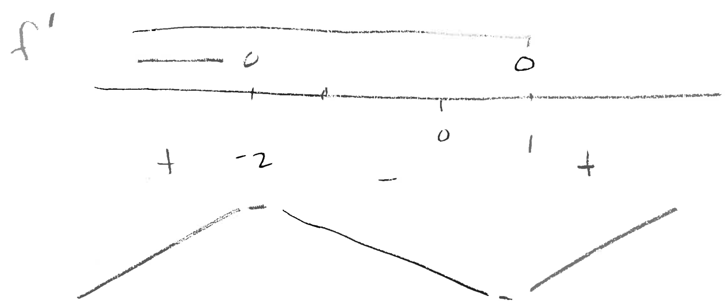
Math 32	**TUFTS UNIVERSITY**	**April 11, 2016**
Calculus I	**Department of Mathematics**	**12-1:20 pm**
All sections	**Exam II**	

No books, notes, **or calculators. TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. If you need to use L'Hôpital's Rule to compute a limit, please indicate that you are using it and note the "form" that applies, e.g. $\frac{0}{0}$. You are required to **sign** your exam book. With your signature, you pledge that you have neither given nor received assistance on this exam.

Problem	Point Value	Points
1	12	
2	6	
3	8	
4	8	
5	8	
6	10	
7	6	
8	8	
9	8	
10	10	
11	8	
12	8	
	100	

1. (12 pts) Given each of the following functions and their derivatives, determine the intervals on which $f(x)$ is increasing and decreasing and the x-values of any local maxima or minima.

(a) $f(x) = 2x^3 + 3x^2 - 12x$ $\qquad f'(x) = 6(x-1)(x+2)$.



f increasing $(-\infty, -2), (1, \infty)$

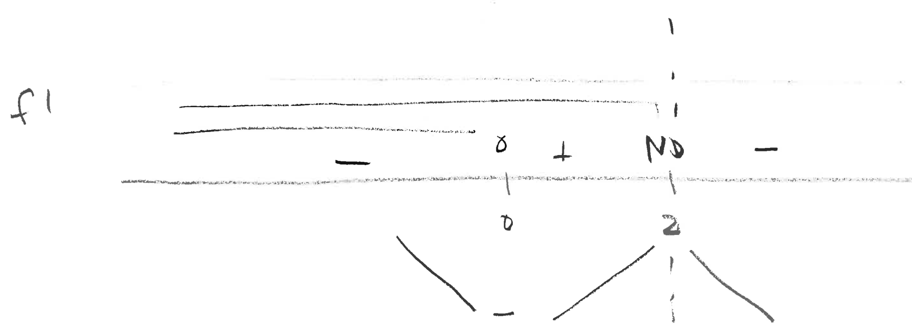
f decreasing $(-2, 1)$

local max at $x = -2$

local min at $x = 1$

(b) $f(x) = \dfrac{x^2}{(x-2)^2}$ $\qquad f'(x) = -\dfrac{4x}{(x-2)^3}$

f(x) is not defined (ND) at $x = 2$



f increasing $(0, 2)$

f decreasing $(-\infty, 0), (2, \infty)$

local min at $x = 0$

there is no local max

2. (6 pts) Given the following function and its second derivative, determine the intervals on which $f(x)$ is concave up (CU) and concave down (CD). List the $x-$values of any points of inflection. Note that $f(x)$ is continuous for all x.

$$f(x) = 5x^{\frac{2}{3}} + x^{\frac{5}{3}} \qquad f''(x) = \frac{10(x-1)}{9x^{\frac{4}{3}}}$$

$f''(x)$ does not exist (d.n.e.) at $x = 0$

f'' −   d.n.e.   −   0   +

0   1

$f(x)$ is CU on $(1, \infty)$

$f(x)$ is CD on $(-\infty, 0), (0, 1)$

There is a point of inflection at $x = 1$.

3. (8 pts) Find the absolute maximum value and absolute minimum value of the function $f(x) = x^3 - 3x^2 - 9x$ on the interval $[-2, 5]$. Show work to justify your answers.

$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

Critical numbers: $f'(x) = 0 = 3(x-3)(x+1)$

$x = 3, -1$

x	$f(x) = x^3 - 3x^2 - 9x$
-2	$-8 - 12 + 18 = -2$
-1	$-1 - 3 + 9 = 5$
3	$27 - 27 - 27 = -27$
5	$125 - 75 - 45 = 5$

absolute max is 5 - it occurs at $x = -1$ and $x = 5$

absolute min is -27 - it occurs at $x = 3$

4. (8 pts)

(a) Find $\lim\limits_{x\to 0^+} x \ln x$

$x \to 0$, $\ln x \to -\infty$

$$= \lim_{x\to 0^+} \frac{\ln x \to -\infty}{\frac{1}{x} \to \infty} \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2}{x}$$

$$= \lim_{x\to 0^+} (-x) = 0$$

(b) Let $f(x) = x \ln x$ on the interval $(0, \infty)$. Then $f(x)$ is continuous on its domain. Find all critical points for $f(x)$ on this domain. Determine whether or not any local extreme values occur at these point(s). If so, can you conclude that any absolute extreme value(s) occur at these point(s)? If so, why?

$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$

critical points: $f'(x) = 0 = \ln x + 1$

$0 = \ln x + 1$

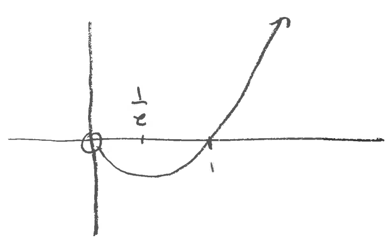
$\ln x = 1 \iff e^{-1} = x$ or $x = \frac{1}{e}$

apply the 2nd derivative test: $f''(x) = \frac{1}{x}$

$f''\left(\frac{1}{e}\right) = e > 0$ (CU)

thus a local min occurs at $x = \frac{1}{e}$. Since it is the only critical point on $(0, \infty)$, an absolute min occurs at $x = 1/e$

(FYI - graph of $y = x \ln x$ on $(0, \infty)$



)

5. (8 pts) Be sure to justify your answers to the following.

(a) Let $f(x) = \dfrac{x^2}{(x-2)^2}$. Find all vertical aymptotes of $f(x)$ or state that none exist.

only candidate: $x=2$

$$\lim_{x \to 2^+} \frac{x^2 \to 4}{(x-2)^2 \to 0(+)} \doteq +\infty$$

thus there is a vertical asymptote: $x=2$

(b) Let $f(x) = \dfrac{2e^{3x}}{50x + e^{3x}}$. Find all horizontal aymptotes of $f(x)$ or state that none exist.

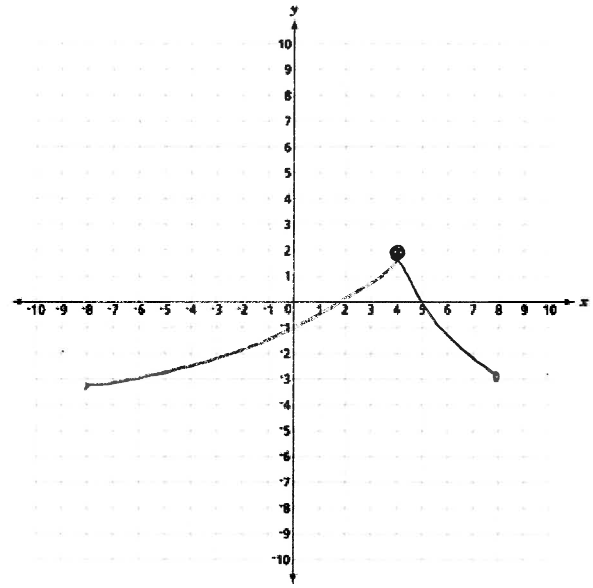
(1) $\lim_{x \to \infty} \dfrac{2e^{3x} \to \infty}{50x + e^{3x} \to \infty} \overset{L}{=} \lim_{x\to\infty} \dfrac{6e^{3x} \to \infty}{50 + 3e^{3x} \to \infty}$

$\overset{L}{=} \lim_{x\to\infty} \dfrac{18e^{3x}}{9e^{3x}} = \lim_{x\to\infty} 2 = 2$ HA $y=2$

(2) $\lim_{x \to -\infty} \dfrac{2e^{3x} \to 0}{50x + e^{3x} \to \infty} = 0$ HA: $y=0$

6. (10 pts) In each case draw the graph of a function which satisfies the given information.

(a) Let $f(x)$ be a continuous function defined on the interval [-8,8] such that $f(4) = 2$.
$f'(x) > 0$ on $(-8, 4)$ and $f'(x) < 0$ on $(4, 8)$; $f'(4)$ does not exist.
$f''(x) > 0$ on $(-8, 4)$ and $f''(x) > 0$ on $(4, 8)$; $f''(4)$ does not exist.



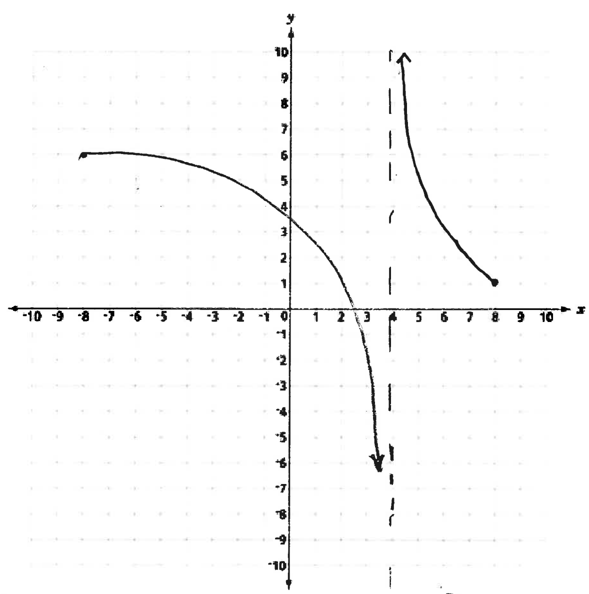
(one possibility)

f' + dne −
4

f'' + dne +
4
CU CU

(b) Let $f(x)$ be a continuous function defined on the intervals [-8,4) and (4,8].
There is a vertical asymptote at $x = 4$.
$f'(x) < 0$ on $(-8, 4)$ and $f'(x) < 0$ on $(4, 8)$
$f''(x) < 0$ on $(-8, 4)$ and $f''(x) > 0$ on $(4, 8)$

* f is not defined (ND) at x=4



(one possibility)

f' − ND −
4

f'' ND
4
CD CU

7. (6 pts) Use linear approximation to estimate $\sqrt{15}$. Choose a value of a that produces a small error. Simplify your answer.

Choose $a=16$ since 16 is close to 15 and we know $\sqrt{16}=4$

$f(x)=\sqrt{x}$, $a=16$. $L(x)=f(a)+f'(a)(x-a)$

$f(a)=f(16)=\sqrt{16}=4$ $\boxed{L(x)=4+\frac{1}{8}(x-16)}$

$f'(x)=\frac{1}{2\sqrt{x}}$ so $f'(16)=\frac{1}{2\sqrt{16}}=\frac{1}{8}$ $\therefore \sqrt{15}\approx L(15)=4+\frac{1}{8}(15-16)$

$=4-\frac{1}{8}=\boxed{3\,7/8 \text{ or } 3.875}$

8. (8 pts) Let $f(x)=\sqrt{x}-\dfrac{x}{3}$ be defined on the interval [0,9].

(a) Then $f(x)$ satisfies the conditions required to apply the Mean Value Theorem. What are those conditions in this case?

f is continuous on $[0,9]$

f is differentiable on $(0,9)$.

(b) Find all points $x=c$ that are guaranteed to exist by the Mean Value Theorem.

Solve for c: $f'(c)=\dfrac{f(9)-f(0)}{9-0}$

$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{3}$

$f'(c)=\dfrac{\left[\sqrt{9}-9/3\right]-0}{9}$

$\frac{1}{2\sqrt{c}}-\frac{1}{3}=0$

$\frac{1}{2\sqrt{c}}=\frac{1}{3}$

$2\sqrt{c}=3$

$\sqrt{c}=\frac{3}{2} \rightarrow c=\frac{9}{4}$ which is in $(0,9)$

9. (8 pts) Find the derivatives of the following functions.

(a) $f(x) = \tan^{-1}(e^{2x})$

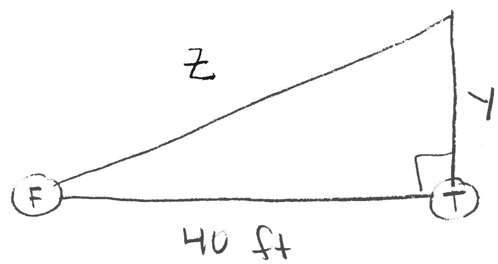
$$f'(x) = \frac{1}{1+(e^{2x})^2} \cdot 2e^{2x}$$

(b) $f(x) = x\sin^{-1}\left(\frac{1}{x}\right)$

$$f'(x) = 1 \cdot \sin^{-1}\left(\frac{1}{x}\right) + x\left[\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\right]\left[-\frac{1}{x^2}\right]$$

10. (10 pts) A flounder lies on the deep sea floor watching a turtle that is 40 feet away and also on the sea floor. The turtle ascends straight up from the sea floor at a rate of 2 ft/sec. How fast is the distance between the flounder and the turtle changing when the turtle is 30 feet above the sea floor? (YOU WILL SET THIS PROBLEM UP AND THAT IS ALL.)

(a) Draw a picture and label label any given fixed values. Label the distance of the turtle from the sea floor with the variable y. Label the distance of the flounder to the turtle with the variable z.



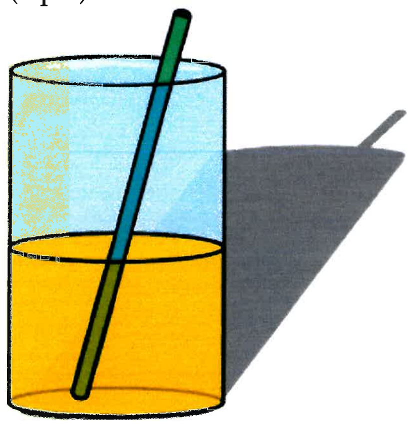
(b) Using your picture write down an equation relating all of the variables.

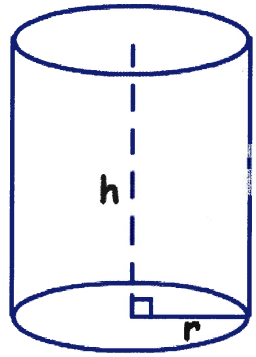
$40^2 + y^2 = z^2$

(c) Using your picture, variable names, and mathematical notation, write down what you have been given and what you have been asked to find and when.

Find $\frac{dz}{dt}$ when $y = 30$ft, given $\frac{dy}{dt} = 2$ ft/sec

11. (8 pts)





Suppose you are drinking juice out of a cylindrical glass that is 6 inches tall and has a fixed radius of 2 inches so that the height of the juice decreases at a rate of 2 in/s. At what rate in the units $(\text{in})^3/\text{s}$ is the juice being drained out of the glass?

The volume V of juice = (area of the base of the cylinder) × (height h of the juice.)

$$V = (\pi r^2) \times (h)$$

$$V = \pi r^2 h \qquad r = 2 \text{ in}$$

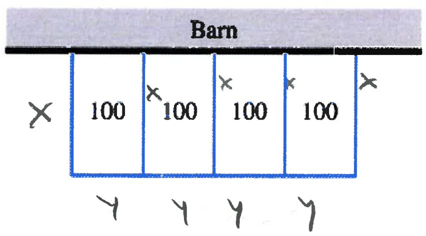
$$V = 4\pi h$$

$$\frac{dV}{dt} = 4\pi \frac{dh}{dt}$$

$$\frac{dV}{dt} = 4\pi \left[-2\right] = -8\pi \text{ in}^3/\text{sec}$$

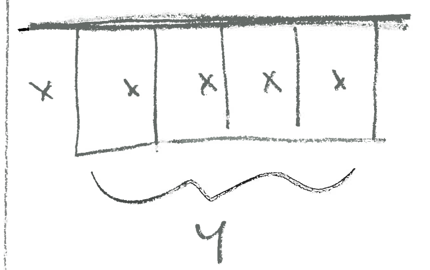
12. (8 pts) A shepherdess is building 4 identical and adjacent rectangular pens against her barn, each with an area of 100 m^2. She wants to minimize the amount of fencing needed. (You will be asked to set this problem up but not solve it.)

(a) Label the diagram with the variables needed to compute the amount of fencing required. Write down an expression for the amount of fencing needed using your variables. In other words, write down the objective function.



$f = 5x + 4y$

OR



$f = 5x + y$

(b) Write down any relationship(s) among the variables.

Given: $xy = 100$ $\left(\text{so } y = \frac{100}{x}\right)$

$xy = 400$

(c) Use the relationship(s) among the variables to eliminate all but one variable of the objective function.

$f(x) = 5x + 4\left[\frac{100}{x}\right] = 5x + \frac{400}{x}$

$f(x) = 5x + \frac{400}{x}$

(d) With the objective function expressed in terms of a single variable, find the interval of interest for that variable.

Since $xy = 100$, x is in $(0, +\infty)$

x is in $(0, \infty)$