Exam 2 Solutions

#1: **(a)** $\sin^{-1}(\sin(7\pi/6)) = \sin^{-1}(-1/2) = -\pi/6$

(b) $\tan^{-1}(-\sqrt{3}) = -\pi/3$

(c) $\tan\left(\cos^{-1}(x)\right) = \dfrac{\sqrt{1-x^2}}{x}$

#2: **(a)** Let $y = \ln(x)/\sin^{-1}(x)$. Then,

$$\frac{dy}{dx} = \frac{\sin^{-1}(x)\left(\frac{1}{x}\right) - \ln(x)\left(1/\sqrt{1-x^2}\right)}{[\sin^{-1}(x)]^2}$$

(b) Let $f(t) = \tan^{-1}(e^{3t})$. Then,

$$f'(t) = \left(\frac{1}{1+(e^{3t})^2}\right)(3)e^{3t}$$

(c) Let $g(x) = \log_3(\cos x)$. Then,

$$g'(x) = \left(\frac{1}{\cos(x)\ln(3)}\right)(-\sin x)$$

(d) Let $h(x) = (1/x)^{\ln x} = x^{-\ln x}$ and $y = h(x)$. Then, $\ln y = -[\ln x]^2$. So,

$$\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = -2(\ln x)(1/x) \quad \Rightarrow \quad h'(x) = \left(x^{-\ln x}\right)\left(\frac{-2\ln x}{x}\right)$$

#3: **(a)** Let $f(x) = -x^{4/5}$, where $-1 \le x \le 32$. Then $f'(x) = (-4/5)x^{-1/5} \neq 0$ for all $x \neq 0$ and $f'(0)$ does not exist. So, the critical points and endpoints are -1, 0, 32. Notice that

$$f(32) = -16 < f(-1) = -1 < f(0) = 0$$

Hence, the point $(32, -16)$ is an absolute minimum, and the point $(0, 0)$ is an absolute maximum.

(b) Let $g(x) = \dfrac{x}{x^2+1}$ where $0 \le x \le 3$. Then,

$$g'(x) = \frac{1-x^2}{(x^2+1)^2} = 0 \quad \text{when } x = -1, 1$$

Since g' exists everywhere, and -1 is not in $[0, 3]$, the critical points and endpoints are 0, 1, 3. Notice that

$$g(0) = 0 < g(3) = 3/10 < g(1) = 1/2$$

Hence, the point $(0, 0)$ is an absolute minimum, and the point $(1, 1/2)$ is an absolute maximum.

#4: Implicitly differentiating $2x^3y + x^2 = 4e^y$ with respect to x gives

$$2x^3y' + y[6x^2] + 2x = 4e^y y' \quad \Rightarrow \quad y'[4e^y - 2x^3] = 6x^2y + 2x$$
$$\Rightarrow \quad y' = \frac{6x^2y + 2x}{4e^y - 2x^3}$$

When $x = 2$ and $y = 0$, we find that $y' = \frac{6(2)^2(0)+2(2)}{4e^0-2(2)^3} = -1/3$. Hence the equation of the tangent line through the point $(2, 0)$ is $y = (-1/3)(x - 2)$.

#5: **(a)** ..., there exists $c \in (-1, 2)$ such that $f'(c) = \dfrac{f(2) - f(-1)}{2 - (-1)} = 5$.

(b) The graph of the following function would suffice:

$$f(x) = \begin{cases} x & \text{if } 1 \leq x < 3 \\ 0 & \text{if } x = 3 \end{cases}$$

#6: Let $f(x) = x + 2\cos(x)$, where $0 \leq x \leq \pi$. Then,

$$f'(x) = 1 - 2\sin(x) \qquad \text{and} \qquad f''(x) = -2\cos(x)$$

Now we find that

$$f'(x) < 0 \quad \Rightarrow \quad \sin x > 1/2 \quad \Rightarrow \quad \pi/6 < x < 5\pi/6 \quad \text{and}$$
$$f'(x) > 0 \quad \Rightarrow \quad \sin x < 1/2 \quad \Rightarrow \quad 0 < x < \pi/6 \quad \text{or} \quad 5\pi/6 < x < \pi$$

Hence, f is decreasing on $(\pi/6, 5\pi/6)$, and increasing on $(0, \pi/6)$ and $(5\pi/6, \pi)$.

$$f''(x) < 0 \quad \Rightarrow \quad \cos x > 0 \quad \Rightarrow \quad 0 < x < \pi/2 \quad \text{and}$$
$$f''(x) > 0 \quad \Rightarrow \quad \cos x < 0 \quad \Rightarrow \quad \pi/2 < x < \pi$$

Hence, f is concave up on $(\pi/2, \pi)$ and concave down on $(0, \pi/2)$.

#7: Let y be the height of the flying saucer, and x be the distance between the turtle and the saucer. Then,

$$\frac{dy}{dt} = -20 \text{ ft/min} \quad \text{and} \quad x^2 = 40^2 + y^2 \quad \Rightarrow \quad 2x\frac{dx}{dt} = 2y\frac{dy}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{-20y}{x}$$

When $y = 30$, we find that

$$x = \sqrt{40^2 + 30^2} = 50 \quad \text{and} \quad \frac{dx}{dt} = \frac{-20(30)}{50} = -12 \text{ ft/min}$$

#8: Given

$$f(x) = 3x^5 - 20x^3 + 7 \qquad f'(x) = 15x^4 - 60x^2 = 15x^2(x-2)(x+2) \qquad f''(x) = 60x^3 - 120x = 60x(x^2-2)$$

Since f' exists everywhere and $f'(x) = 0$ when $x = -2, 0, 2$, we can use the following sign diagram:

Interval	$15x^2$	$x-2$	$x+2$	f'
$x < -2$	$+$	$-$	$-$	$+$
$-2 < x < 0$	$+$	$-$	$+$	$-$
$0 < x < 2$	$+$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$	$+$

Since f'' exists everywhere and $f''(x) = 0$ when $x = -\sqrt{2}, 0, \sqrt{2}$, we can use the following sign diagram:

Interval	$60x$	x^2-2	f''
$x < -\sqrt{2}$	$-$	$+$	$-$
$-\sqrt{2} < x < 0$	$-$	$-$	$+$
$0 < x < \sqrt{2}$	$+$	$-$	$-$
$\sqrt{2} < x$	$+$	$+$	$+$

(a) f is increasing on $(-\infty, -2)$ and $(2, \infty)$.

(b) f is decreasing on $(-2, 0)$ and $(0, 2)$.

(c) f' changes from positive to negative at -2, so f has a local maximum at $x = -2$.

(d) f' changes from negative to positive at 2, so f has a local minimum at $x = 2$.

(e) The graph of f is concave up on $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$.

(f) The graph of f is concave down on $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$.

(g) The x coordinates of the inflection points are $-\sqrt{2}$, 0, and $\sqrt{2}$.