Math 32

Tufts University
Department of Mathematics
Exam 2 Review Solutions

Spring 2014

1.

(a) $y = \cos^{-1}(\sqrt{1+x^2} - x)$ Note: $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

$$\boxed{y' = \frac{-1}{\sqrt{1-(\sqrt{1+x^2}-x)^2}} \cdot (\frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x - 1)}$$

(b) $y = \ln x + 2^e + 5^x + x^7 + \ln x^{1/2} + \ln 8 + e^{4x}$ Note: $5^x = e^{x \ln 5}$ and $\ln x^{1/2} = \frac{1}{2}\ln x$

$$\boxed{y' = \frac{1}{x} + (\ln 5)5^x + 7x^6 + \frac{1}{2x} + 4e^{4x}}$$

(c) $y = \tan^{-1}(\sin(x^2) - \cos(x^2))$ Note: $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$$\boxed{y' = \frac{1}{1+(\sin x^2 - \cos x^2)^2} \cdot [2x\cos(x^2) + 2x\sin(x^2)]}$$

(d) $y = (5x+2)^{\cos x}$ Use logarithmic differentiation: $\ln y = \cos x \ln(5x+2)$

Now we take the derivative: $\frac{y'}{y} = (\cos x) \cdot \frac{5}{5x+2} + \ln(5x+2) \cdot (-\sin x)$

$$y' = \left[\frac{5\cos x}{5x+2} - \sin x \ln(5x+2)\right] y \qquad \boxed{y' = \left[\frac{5\cos x}{5x+2} - \sin x \ln(5x+2)\right] \cdot (5x+2)^{\cos x}}$$

(e) $\frac{(7x-2)^4 \cdot (x^2+5)^{17}}{(4x^3+9x)^6}$. Use logarithm differentiation:

$$\ln y = \ln[(7x-2)^4 \cdot (x^2+5)^{17}] - \ln[(4x^3+9x)^6] = 4\ln(7x-2) + 17\ln(x^2+5) - 6\ln(4x^3+9x)$$

Now we take the derivative. $\frac{y'}{y} = 4 \cdot \frac{7}{7x-2} + 17 \cdot \frac{2x}{x^2+5} - 6 \cdot \frac{(12x^2+9)}{4x^3+9x}$

$$y' = \left[\frac{28}{7x-2} + \frac{34}{x^2+5} - \frac{6(12x^2+9)}{4x^3+9x}\right] \cdot y \qquad \boxed{y' = \left[\frac{28}{7x-2} + \frac{34}{x^2+5} - \frac{6(12x^2+9)}{4x^3+9x}\right] \cdot \frac{(7x-2)^4 \cdot (x^2+5)^{17}}{(4x^3+9x)^6}}$$

2.

(a) $g(x) = \frac{\sqrt{e^x+1}}{\ln(x+1)-1}$ Note: $e^x > 0$ for all x. So $e^x + 1 > 1$ for all x.

Need: $x + 1 \geq 0$ and $\ln(x+1) - 1 \neq 0$

Where is $\ln(x+1) - 1 = 0$? $\ln(x+1) = 1$ $x + 1 = e$ $x = e - 1$

The domain is $\boxed{(1, e-1) \cup (e-1, \infty)}$

(b) $h(x) = \ln[(x+3)(x+1)(2-\sqrt{x})]$ $= \ln(x+3) + \ln(x+1) + \ln(2-\sqrt{x})$

Need; $x+3 > 0$ and $x+1 > 0$ and $2-\sqrt{x} > 0$. Also need $x > 0$ (because the function contains $\sqrt{x}$).

$2 - \sqrt{x} > 0$ implies $2 > \sqrt{x}$ implies $0 \leq x < 4$.

So we need $x > -3$ and $x > -1$ and $x < 4$ and $x > 0$. So the domain is $\boxed{(0,4)}$

3. Critical points are interior points to an interval where the derivative is 0 or does not exist.

(a) $f(x) = \sqrt[3]{x^2 - x} = (x^2 - x)^{\frac{1}{3}}$ Domain of $f(x)$: all real numbers (odd root).

$$f'(x) = \frac{1}{3}(x^2 - x)^{-\frac{2}{3}}(2x - 1) = \frac{2x - 1}{3(x^2 - x)^{\frac{2}{3}}}$$

Find the domain of $f'(x)$. $x(x - 1) = 0$ when $x = 0, x = 1$.

Domain of $f'(x) : (-\infty, 0) \cup (0, 1) \cup (1, \infty)$qquad So $f'(0)$ DNE and $f'(1)$ DNE.

Solve $f'(x) = 0$: $f'(x) = 0$ when $2x - 1 = 0, x = \frac{1}{2}$ The critical points are at $\boxed{x = 0, \frac{1}{2}, 1}$

(b) $f(x) = \sin^2 x - x$ Domain: all real numbers.

$f'(x) = 2\sin x \cos x - 1$ Solve $f'(x) = 0$:

$2\sin x \cos x = 1$ $\sin(2x) = 1$ $2x = \frac{\pi}{2} + 2n\pi$, n an integer $x = (\frac{\pi}{2} + 2n\pi) \cdot \frac{1}{2}$

Critical points: $\boxed{x = \frac{\pi}{4} + n\pi, \text{n an integer}}$

(c) $f(x) = |2x-1| = \begin{cases} 2x - 1 & \text{if } 2x - 1 \geq 0 \\ -(2x - 1) & \text{if } 2x - 1 < 0 \end{cases}$ $f'(\frac{1}{2})$ DNE $\boxed{x = \frac{1}{2} \text{ is the only critical number}}$

4.

i) The Extreme Value Theorem says that a continuous function on a closed interval has an absolute maximum and an absolute minimum.

ii) The candidates for the absolute extrema are critical points (points interior to the interval where the derivative is 0 or does not exist) and the endpoints of the interval.

iii)

(a) $f(x) = x^3 - 6x^2 + 9x + 1$ on $[2, 4]$

$f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0$: $3(x^2 - 4x + 3) = 0$

$(x - 1)(x - 3) = 0$

$x = \underbrace{1}_{\text{not in the interval}}, 3$

x	$f(x)$
2	$f(2) = 8 - 24 + 18 + 1 = 3$
3	$f(3) = 27 - 54 + 27 + 1 = 1$ min
4	$f(4) = 64 - 96 + 36 + 1 = 5$ max

$\boxed{\text{abs.min f(3)} = 1 \qquad \text{abs.max f(4)} = 5}$

(b) $g(x) = \dfrac{x^2 - 9}{x^2 + 9}$ on [1,4]

$$g'(x) = \frac{(x^2 + 9)(2x) - (x^2 - 9)(2x)}{(x^2 + 9)^2} = \frac{2x^3 + 18x - 2x^3 + 18x}{(x^2 + 9)^2} = \frac{36x}{(x^2 + 9)^2}$$

$g'(x) = 0$ when $x = \underbrace{0}_{\text{not in the interval}}$

x	$f(x)$
1	$f(1) = -\frac{8}{10} = -\frac{4}{5}$ min
4	$f(4) = \frac{7}{25}$ max

$\boxed{\text{abs.min f(1)} = -\frac{4}{5} \qquad \text{abs.max f(4)} = \frac{7}{25}}$

(c) $h(x) = \sqrt[3]{x}(8-x)$ on $[0,8]$

$h'(x) = \sqrt[3]{x}(-1) + (8-x)\frac{1}{3}x^{-\frac{2}{3}}$

$\underline{h'(x) = 0}:$ $\quad \dfrac{8-x}{3x^{\frac{2}{3}}} = x^{\frac{1}{3}}$

$8 - x = 3x \qquad 8 = 4x \qquad x = 2$

x	$f(x)$
0	0 min
2	$6\sqrt[3]{2}$ max
8	0 max

abs.min f(0) = f(8) = 0 abs.max f(2) = $6\sqrt[3]{2}$

(d) $f(x) = e^{-x} - e^{-2x}$ on $[0, \ln 9]$ $\qquad f'(x) = -e^{-x} + 2e^{-2x}$

$\underline{f'(x) = 0}: 2e^{-2x} = e^{-x} \qquad 2 = \frac{e^{-x}}{e^{-2x}} = e^{-x} \cdot e^{2x} = e^{x} \qquad x = \ln 2$

x	$f(x)$
0	0
$\ln 2$	$e^{-\ln 2} - e^{-2\ln 2} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$
$\ln 9$	$e^{-\ln 9} - e^{-2\ln 9} = \frac{1}{9} - \frac{1}{81} = \frac{8}{81}$

abs.min f(0) = 0 abs.max f(ln 2) = $\dfrac{1}{4}$

(e) $g(x) = \dfrac{\ln x}{x^2}$ on $[1, e]$ $\qquad g'(x) = \dfrac{x^2 \cdot \frac{1}{x} - (\ln x)(2x)}{x^4} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3}$

$\underline{g'(x) = 0}$: where $1 - 2\ln x = 0 \qquad 1 = 2\ln x \qquad \ln x = \frac{1}{2} \qquad x = e^{\frac{1}{2}}$

x	$g(x)$
1	$f(1) = \frac{\ln 1}{1} = 0$
$e^{\frac{1}{2}}$	$f(e^{\frac{1}{2}}) = \frac{\ln e^{\frac{1}{2}}}{e} = \frac{1}{2e}$
e	$f(e) = \frac{\ln e}{e^2} = \frac{1}{e^2}$

Since $2 > e$, so $2e < e^2$, $\frac{1}{2e} > \frac{1}{e^2}$: abs.min $f(1) = 0$ abs.max $f(e^{\frac{1}{2}}) = \dfrac{1}{2e}$

5. (a) $f(x) = 3x^3 - 18x^2 + 5$ $f'(x) = 9x^2 - 36x$ Domain: all real numbers.

$\underline{f'(x) = 0}: 9x(x-4) = 0 \qquad x = 0, 4;$

Using a sign diagram: $f'(x) > 0$ on $(-\infty, 0), (4, \infty)$, hence, f increases there;

$f'(x) < 0$ on $(0, 4)$, hence, f decreases there. So: local min at x = 4 local max at x = 0

(b) $f(x) = \dfrac{x}{x^2 - 9} = \dfrac{x}{(x-3)(x+3)}$ $\qquad$ Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

$f'(x) = \dfrac{(x^2-9)(1) - x(2x)}{(x^2-9)^2} = \dfrac{x^2 - 9 - 2x^2}{(x-9)^2} = \dfrac{-9 - x^2}{(x^2-9)^2} = -\dfrac{(9 + x^2)}{(x^2-9)^2}$

$\underline{f'(x) = 0}$: never since $f(x) < 0$ for all x in its domain. f has no local min or local max

(c) $f(x) = \dfrac{x^2}{x^2 + 9}$ $\qquad$ Domain: all real numbers.

$f'(x) = \dfrac{(x^2+9)(2x) - x^2(2x)}{(x^2+9)^2} = \dfrac{2x^3 + 19x - 2x^3}{(x^2+9)^2} = \dfrac{18x}{(x^2+9)^2}$

$\underline{f'(x) = 0}$: where $x = 0$; $f'(x) > 0$ on $(0, \infty)$ hence f increases; $f'(x) < 0$ on $(-\infty, 0)$, hence f decreases.

local min at x = 0

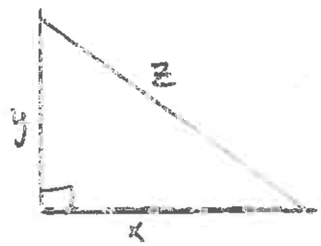
(d) $f(x) = \sin x - \cos x$ in $[0, 2\pi]$; $f'(x) = \cos x + \sin x$ Since $f(x)$ is continuous and differentiable on $(0, 2\pi)$, it will only change where it increases or decreases when $f'(x) = 0$.

$\underline{f'(x) = 0}: \cos x + \sin x = 0 \quad \sin x = -\cos x \quad \frac{\sin x}{\cos x} = -1 \quad \tan x = -1 \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f'(\frac{\pi}{2}) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1, \quad f'(\pi) = \cos\pi + \sin\pi = -1, \quad f'(\frac{11\pi}{6}) = \cos\frac{11\pi}{6} + \sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} > 0$

So f increases on $(0, 3\pi/4)$, decreases on $(3\pi/4, 7\pi/4)$, increases on $(7\pi/4, 2\pi)$, hence

$\boxed{\text{local mac at x} = \frac{3\pi}{4} \qquad \text{local min at x} = \frac{7\pi}{4}}$



6. $\underline{\text{Given}}$: $\frac{dx}{dt} = 25\text{mph}$; $\frac{dy}{dt} = 35\text{mph}$ $\underline{\text{Find}}$: $\frac{dz}{dt}|_{t=4}$

Let $z =$ the distance between the two cars.

$z = \sqrt{x^2 + y^2}; \quad \frac{dz}{dt} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot (2x\frac{dx}{dt} + 2y\frac{dy}{dt})$

Now plug in x and y when $t = 4$ hours.

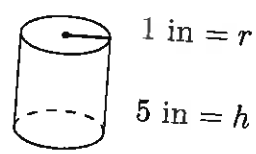
$x = \frac{dx}{dt} \cdot t = \frac{25 \text{ miles}}{\text{hr}} \cdot 4 \text{ hrs} = 100 \text{ miles} \qquad x = \frac{dy}{dt} \cdot t = \frac{35 \text{ miles}}{\text{hr}} \cdot 4 \text{ hrs} = 130 \text{ miles}$

So $\frac{dz}{dt}$ when $t = 4$ is $\boxed{\frac{dz}{dt} = \frac{1}{2}(100^2 + 140^2)^{-\frac{1}{2}}(200 \cdot 25 + 280 \cdot 35) \text{ mph} = \frac{14,800}{2\sqrt{29,600}}}$

7. Given: $\frac{dV}{dt} = 10\pi \text{ in}^3/\text{sec}$, $\frac{dr}{dt} = 3$ in/sec Find: $\frac{dh}{dt}$.

$V = \pi r^2 h$: $\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} + h 2\pi r \frac{dr}{dt}$

$10\pi = \pi \cdot \frac{dh}{dt} + 5 \cdot 2\pi \cdot 1 \cdot 3; \quad 10\pi - 30\pi = \pi\frac{dh}{dt}$



$\boxed{\frac{dh}{dt} = -20 \text{ in/sec}}$

8. $h =$ water level, $r =$ radius of water surface, $V =$ volume of water

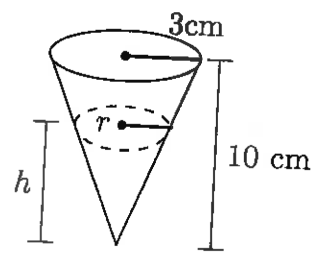
Given: $\frac{dV}{dt} = 2\text{cm}^3/\text{sec}$; Find: $\frac{dh}{dt}|_{h=5\,cm}$

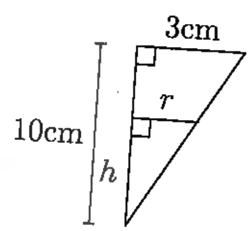
By similar triangles, $\frac{r}{3} = \frac{h}{10}$; $r = \frac{3}{10}h$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{3}{10}h)^2 \cdot h = \frac{3}{100}\pi h^3$

$\frac{dV}{dt} = \frac{3\pi}{100} \cdot 3h^2 \cdot \frac{dh}{dt}; \quad 2 = \frac{9\pi}{100} \cdot 25 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{200}{25(9\pi)} = \boxed{\frac{8}{9\pi} \text{ cm/sec}}$



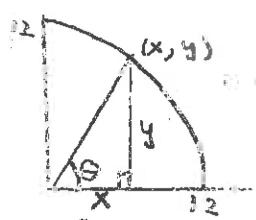


9. Given: $dy/dt = -1$ foot/min Find: $d\theta/dt|_{y=6 \text{ feet}}$

$y = 12\sin\theta \qquad dy/dt = 12\cos\theta \; d\theta/dt$

When $6 = y = 12\sin\theta$, $\sin\theta = 6/12 = 1/2$ so $\theta = \pi/6$

So $d\theta/dt|_{y=6} = \frac{dy/dt}{12\cos(\pi/6)} = \frac{-1}{12(\sqrt{3}/2)} = \boxed{-\frac{1}{6\sqrt{3}} \text{ rad/min}}$



10. (a) $\lim_{x\to\infty} x^{-3}e^x =^{0\cdot\infty} \lim_{x\to\infty} \frac{e^x}{x^3} =^{\infty/\infty} \lim_{x\to\infty} \frac{e^x}{3x^2} =^{\infty/\infty} \lim_{x\to\infty} \frac{e^x}{6x} =^{\infty/\infty} \lim_{x\to\infty} \frac{e^x}{6} = \boxed{\infty}$

(b) $= \lim_{x\to\infty} \frac{\ln\ln x}{x} =^{\infty/\infty} \lim_{x\to\infty} \frac{1}{x\ln x} = \boxed{0}$ (c) $\lim_{x\to 1} \frac{\ln x}{\sin\pi x} =^{0/0} \lim_{x\to 1} \frac{\frac{1}{x}}{\pi\cos\pi x} = \boxed{\frac{1}{-\pi}}$

(d) $\lim\limits_{x\to 0} \dfrac{1-e^{x}}{\cos x} = \dfrac{0}{1} = \boxed{0}$

(e) $\lim\limits_{x\to 0^+} (\tan x)^x =^{0^0} \lim\limits_{x\to 0^+} e^{x\ln(\tan 2x)} = e^L = e^0 = \boxed{1}$ since

$$L = \lim_{x\to 0^+} x\ln(\tan x) =^{0\cdot(-\infty)} \lim_{x\to 0^+} \frac{\ln(\tan x)}{\frac{1}{x}} =^{-\infty/\infty} \lim_{x\to 0^+} \frac{\frac{\sec^2 x}{\tan 2x}}{-x^{-2}} = \lim_{x\to 0^+} \frac{-x^2\sec^2 x}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{-x^2\cdot 2\sec x\cdot \sec x\tan x - 2x\sec^2 x}{\sec^2 x} = 0/2 = 0$$

(f) $\lim\limits_{x\to\infty} \left(1+\dfrac{4}{x}\right)^x =^{1^\infty} \lim\limits_{x\to\infty} e^{x\ln(1+\frac{4}{x})} = e^{\lim\limits_{x\to\infty} x\ln(1+\frac{4}{x})} = e^L = \boxed{e^4}$ since

$$L = \lim_{x\to\infty} x\ln(1+\frac{4}{x}) =^{0\cdot\infty} \lim_{x\to\infty} \frac{\ln(1+4x^{-1})}{\frac{1}{x}} =^{0/0} \lim_{x\to\infty} \frac{-4x^{-2}}{1+4x^{-1}}\cdot(-x^2) = \lim_{x\to\infty} \frac{4}{1+4x^{-1}} = 4$$

(g) $\lim\limits_{x\to 0^+} \dfrac{1}{x} + \ln x =^{\infty-\infty} \lim\limits_{x\to 0^+} \dfrac{1+x\ln x}{x} = \dfrac{1+\lim\limits_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}}{\lim\limits_{x\to 0^+} x}$ (limit in numerator is $-\infty/\infty$)

$$= \frac{1+\lim\limits_{x\to 0^+}\frac{1}{x}\cdot(-x^2)}{\lim\limits_{x\to 0^+} x} = \frac{1+\lim\limits_{x\to 0^+}(-x)}{\lim\limits_{x\to 0^+} x} = \lim_{x\to 0^+}\frac{1-x}{x} = \lim_{x\to 0^+}\frac{1}{x} - 1 = \boxed{\infty}$$

11. $f(x) = \dfrac{18(x+1)}{(x+3)^2}$ Domain: $(-\infty,3)\cup(-3,\infty)$

(a) $f(0) = \dfrac{18}{9} = 2$ $\boxed{(0,2)\ y-\text{intercept}}$ $f(x) = \dfrac{18(x+1)}{(x+3)^2} = 0$ $\boxed{(-1,0)\ y-\text{intercept}}$

(b) $\lim\limits_{x\to\infty} \dfrac{18x+18}{x^2+6x+9} = \lim\limits_{x\to\infty} \dfrac{\frac{18}{x}+\frac{18}{x^2}}{1+\frac{6}{x}+\frac{9}{x^2}} = 0$ $\lim\limits_{x\to-\infty} \dfrac{18x+18}{x^2+6x+9} = 0$ $\boxed{y=0 \text{ H.A.}}$

$\lim\limits_{x\to -3^-} \dfrac{18(x)1)}{(x+3)^2} = -\infty$ $\lim\limits_{x\to -3^+} \dfrac{18(x)1)}{(x+3)^2} = -\infty$ $\boxed{x=-3 \text{ V.A.}}$

(c) Solve $\underline{f'(x)=0}$: $\dfrac{18(1-x)}{(x+3)^3} = 0$ $x=1$

$f'<0$ and hence decreases on $(-\infty,-3),(1,\infty)$; $f'>0$ and hence increases on $(-3,1)$

(d) No local min since $x=-3$ is not in the domain. Local max at $x=1$.

$f(1) = \dfrac{36}{16} = \dfrac{18}{8} = \dfrac{9}{4}$ $\boxed{\text{local max at } \left(1,\frac{9}{4}\right)}$

(e) Solve $f''(0)$: $\dfrac{36(x-3)}{(x+3)^4} = 0$; $x=3$; f CD on $(-\infty,-3),(-3,3)$ and CU on $(3,\infty)$

(f) $\boxed{\text{inflection point at } (3,2)}$ since $f(3) = \dfrac{18(4)}{36} = \dfrac{4}{2} = 2$