

# Compensation Handout

# SOLUTIONS

Three math students (named A, B and C) share a book, and when class is over, have to decide who gets it. We'll look at 5 different possible compensation arrangements, and then compare them.

Suppose these are the values each student has for the book:

$$a = \$18$$

$$b = \$24$$

$$c = \$30$$

fair shares      6                      8                      10

1) Suppose C gets the book and pays A and B each \$2. What is C's payout?  $X_C = 30 - 4 = 26$

fair? **NO** not to A or B

envy-free? A thinks C gets  $18 - 4 = 14 > 2$   
B thinks C gets  $24 - 4 = 20 > 2$

equitable? **NO**

Pareto-optimal? **NO. A + B both envy C**

$$\frac{2}{18} \neq \frac{2}{24} \neq \frac{26}{30}$$

**YES** high bid is C

2) Suppose C gets the book and pays A and B each \$8. What is C's payout?  $X_C = 30 - 16 = 14$

fair? **Yes.**

envy-free? A thinks C got  $18 - 16 = 2$   
B thinks C got  $24 - 16 = 8$

equitable? **NO**

Pareto-optimal?

$$\frac{8}{18} \neq \frac{8}{24} \neq \frac{14}{30}$$

**YES** high bid is C

**Yes** envy-free.

3) For an equitable and fair arrangement, who can get the book?  $M = \frac{72}{3} = 24$  **B or C**

Suppose B gets the book. Find an equitable, and fair arrangement.

$$q = \frac{24}{72} = \frac{1}{3}$$

$$X_A = \frac{1}{3}(18) = 6$$

$$X_B = \frac{1}{3}(24) = 8$$

$$X_C = \frac{1}{3}(30) = \frac{10}{24} \checkmark$$

B gets book

is this envy-free? **NO**

Pareto-optimal? **NO** high bid did not win

A + B envy C (also C envies B since C thinks B got  $30 - 16 = 14 > 10$ )

4) Find a **Pareto-optimal, fair, equitable** arrangement. Who gets the book? **C**

$$q_b = \frac{30}{12} = \frac{5}{2}$$

$$X_A = \frac{5}{12}(18) = \frac{5}{2}(3) = \frac{15}{2} = \$7.50$$

$$X_B = \frac{5}{12}(24) = 5(2) = \$10$$

$$X_C = \frac{5}{12}(30) = \frac{5}{2}(5) = \frac{25}{2} = \frac{12.50}{30} \checkmark$$

is this envy free?

no  
A envies B.

is this an objective improvement over arrangement 3?

Yes!

5) Find all Pareto-optimal, fair, and envy-free arrangements:

Pareto-optimal requires... C gets book  
 envy-free requires... equal payouts, no envy of C. So let  $X_A = X_B = X$

A thinks C got  $18-2x$   
 So A won't envy C if  $18-2x \leq x$   
 $18 \leq 3x$   
 $x \geq 6$

B thinks C got  $24-2x$   
 So B won't envy C if  $24-2x \leq x$   
 $24 \leq 3x$   
 $x \geq 8$

both true when  $x \geq 8$  and this also makes it fair to A+B

Fair to C requires  $x \leq 10$

So, C gets book; pays A+B each  $x$  where  $8 \leq x \leq 10$

6) Looking at all the arrangements, there are two categories of better arrangements:

What arrangements were fair, Pareto-optimal, and equitable? **4**

What arrangements were fair, Pareto-optimal, and envy-free? **2, 5**

Is a fair, Pareto-optimal, equitable, envy-free arrangement possible for this book?

No. C would have to get the book and the non-winners then are A+B, who have two different values for the book.