Proofs

Here are four 'proofs' of the statement "If a division is envy free, then it is fair." Can you find what is wrong with each of them?

Solutions below show problem areas in red, and explanations (and sometimes a way to make a good proof out of it) in blue.

1. Since the division is envy free, A in particular has no envy. Since A has no envy, the division must be fair to A. Similarly, B has no envy, so the division must be fair to B. This is true for all the participants, so it is fair to everyone.

   Why is it fair to A if A does not envy anyone? This is the meat of the proof and therefore needs more explanation.

2. Since the division is envy free, A does not want anyone else's slice. In other words, A got a largest slice (to A). This must be fair to A. B also thinks B got a largest slice, so it is fair to B. This is true for all the participants, so it is fair to everyone.

   Why does A getting a largest slice mean it is fair to A? As in the first example, this needs to be proven.

   Here's one way to make a good proof out of this:

   Since the cake is cut into N pieces, at least one slice is worth at least $\frac{1}{N}$ to A.
   A largest slice, therefore, must be worth at least A's fair share.
   Since A does not envy anyone, A gets a largest slice and so the division is fair to A.
   Similarly, B gets a largest slice to B, which must be fair to B for the same reasoning that a largest slice is fair to A.
   This is true for all players, so it is fair to everyone.

3. Proof by contrapositive. The contrapositive of the statement is "If a division is unfair, then there is envy." Since the division is unfair, every candidate gets less than $\frac{1}{N}$ of the cake. But if every candidate gets less than $\frac{1}{N}$, then the sums of the values of the slices is less than 1, a contradiction. So there must be envy.

   First problem: the division being unfair does not mean **every** candidate gets less than $\frac{1}{N}$, it means at least one candidate gets less than $\frac{1}{N}$.

   Second problem: The way values for the slices works is that for each candidate, the sum of the slice values must be 1. So to A, the sum of the slices must be 1. It is not true that whatever A gets + whatever B gets + whatever C gets+...etc must be 1.

   Third problem: What are we contradicting? How are we drawing the conclusion that there must be envy? We never assume the division is envy free, so the conclusion for the contradiction does not make any sense.

4. Proof by contrapositive. Since the division is unfair, it is unfair to A. And since it is unfair to A, then A did not get the largest slice (in A's eyes). Therefore A envies whoever did.

   Why does A it being unfair to A mean A did not get the largest slice? This needs to be proven!

   Here's one way to make a good proof out of this:

Proof by contrapositive:
We're trying to prove that "If a division is unfair, then there is envy."
Since the division is unfair, it is unfair to someone. Call this person A.
We know (as we saw in another proof above) that since the cake is cut into N pieces, at least one slice is worth at least $\frac{1}{N}$ to A.
A largest slice, therefore, must be worth at least A's fair share.
Since the division is unfair to A, then A could not have gotten a largest slice.
Since A did not get a largest slice, then A is envious of whoever did.

5. Bonus Proof!

   Proof by contrapositive and contradiction:
   Since the division is unfair, it is unfair some participant, say A.
   Since it is unfair to A, A gets less than $\frac{1}{N}$.
   Assume, for contradiction, that this division is envy free.
   Then A does not envy any of the other participants.
   In A's eyes, each of them get less than A, in particular they too got less than $\frac{1}{N}$.
   But then the sum of the N slices is worth less than 1 to A! This is a contradiction,
   so our assumption is false and the division must have envy.

6. Another Bonus Proof!

   We assume the division is envy-free. So A, in particular, has no envy.
   There are now two possibilities:
   Either A thinks A got $\frac{1}{N}$, and so did everyone else (as in the equal division), which is fair to A
   OR, A thinks A got $> \frac{1}{N}$, and others all got less than whatever A got which is still fair to A.
   Notice it is not possible for A to think that A got $< \frac{1}{N}$, since in that case someone would have to get more than $\frac{1}{N}$, so that all the pieces everyone got will add up to 1 in A's value system.
   Everything we just said for A can be repeated for any player. So the division is fair.