MATH 19-01: APPORTIONMENT NOTES, WEEK 1

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Apprortionment is dividing up representation, such as seats on a governing body and a population into districts.

If the goal is proportional representation, what would be totally fair is if each district's proportion of representation was equal to its proportion of the population. The problem with this? If we were to actually make these proportions *equal*, we would typically need to give out fractional representation, which doesn't really make sense– what would a franction of a representative look like?

So to get whole number (or integer) representation, we will need to round. Now the question is: how?

Notation. Denote the number of seats by m and the total population by M. Then we have

$$m = m_1 + m_2 + \cdots + m_s$$

and

$$M = M_1 + M_2 + \cdots + M_s$$

Where m_i is the number of seats given to the ith district, M_I is the population of the ith district, and $1 \leq i \leq s$ where s is the totaly number of districts.

The *quota* of seats given to the ith district is

$$Q_i = \frac{M_i}{M} \cdot m = \frac{M_i}{M/m}.$$

This is the number of seats the district would receive with our "totally fair" apportionment method– typically not a whole number.

We will use $\lfloor x \rfloor$ to denote the largest integer smaller than x (rounding down) and $\lceil x \rceil$ to denote the smallest integer greater than x (rounding up). In this setting $\{x\}$ will denote the fractional part of x. For example, if $x = 4.3$, then $\lfloor 4.3 \rfloor = 4$, $\lceil 4.3 \rceil = 5$, and $\{4.3\} = 0.3$.

Hamilton's method. One answer to the question "How do we round?" is Hamilton's method, which works as follows:

First assign to each district $\lfloor Q_i \rfloor$ seats (its quota rounded down. Next, if there are any seats left over, give them out in order of $\{Q_i\}$. Let's look at an example to see how this works.

Example of using Hamilton's method. Let $m = 100$ and $s = 3$. Suppose that $M_1 = 505$, $M_2 = 492$, and $M_3 = 301$. Then $M = 505 + 492 + 301 = 1298$ and

$$Q_1 = \frac{505}{1298} \cdot 100 = 38.906\ldots$$
$$Q_2 = \frac{492}{1298} \cdot 100 = 37.904\ldots$$
$$Q_3 = \frac{301}{1298} \cdot 100 = 23.18\ldots.$$

Then $\lfloor Q_1 \rfloor = 38$, $\lfloor Q_2 \rfloor = 37$, and $\lfloor Q_3 \rfloor = 23$, which means that we initiall allocate $38+37+23 = 98$ seats – 2 are left over. Next we see that

$$\{Q_1\} = 0.906\ldots$$
$$\{Q_2\} = 0.904\ldots$$
$$\{Q_3\} = 0.18\ldots$$

so the first extra seat is give to district 1 and the second to district 2. Thus, $m_1 = 39$, $m_2 = 38$, and $m_3 = 23$, which we can check to see that these add up to 100.

Problems with Hamilton's method. While it may seem like a very reasonable way to deal with the rounding problem, Hamilton's method is not without issue. Some unexpected and (depending on your viewpoint) undesirable things can happen with this method, which we'll call paradoxes. Here are a few:

The Alabama Paradox. In the late 1800's, there were 299 seats in the U.S. House of Representatives ($m = 299$) and Alabama was allocated 8 seats using Hamilton's method ($m_{AL} = 8$). Someone noticed that if one seat was added to the house ($m = 300$) and then seats were reallocated, Alabama would end up losing a seat ($m_{AL} = 7$). This doesn't seem right: if m increases, we would expect that each of the m_i would either stay the same or increase.

The Population Paradox. In 1900, Virginia's population was growing 60% faster than Maine's. However, when seats were redistributed following the census, Virginia lost a seat while Maine gained one. If we had to have chosen one of these states to gain a seat, we probably would have gone with Virginia, but using Hamilton's method, the opposite happened.

The New States Paradox. In 1946, Oklahoma became the 46th state in the union. Before this, we had $m = 386$, and based on states of similar size, $m_{OK} = 5$ was assigned, and those seats were added, making $m' = 391$. Hamilton's method was used to redistribute seats, which led to the following:

$$m_{NY} = 38\ ,\ m'_{NY} = 37\ ;$$
$$m_{ME} = 3\ ,\ m'_{ME} = 4.$$

The addition of the 5 seats cause New York's allocation to decrease and Maine's to increase, almost like Maine recieved one of New York's seats simply because Oklahoma showed up on the scene. No good!

Alternatives to Hamilton's method. The idea behind many alternative apportionment systems is to adjust the denominator of $Q_i = \frac{M_i}{M/m}$ and round (in some way) until there are no leftover seats.

One such alternative is *Jefferson's method*, in which we adjust M/m and round down until, through trial and error, we find the denominator that gives us whole numbers and no leftover seats. (Note, such a denominator will always exist!)

Let's revisit the example we used earlier to demonstrate this new method. We have $M = 1298$ and $m = 100$, so $M/m = 12.98$. Recall that none of the Q_i were integers using this denominator. So what if we divide by 12.9 instead? Then we get

$$
\begin{aligned}
Q_1 &= \lfloor \frac{505}{12.9} \rfloor = 39 \\
Q_2 &= \lfloor \frac{492}{12.9} \rfloor = 38 \\
Q_3 &= \lfloor \frac{301}{12.9} \rfloor = 23
\end{aligned}
$$

so $Q_1 + Q_2 + Q_3 = 100 = m$. No leftover seats!

The quota rule. We will say that an apportionment system satisfies the *quota rule* is

$$\lfloor Q_i \rfloor \leq m_i \leq \lceil Q_i \rceil$$

for all i. That is, every district receives a number of seats that is either its quota rounded down or its quota rounded up.

Note, Hamilton's method satisfies the quota rule: each district is initially assigned $m_i = \lfloor Q_i \rfloor$ seats. If they then recieve an extra seat, they end will end up with $m_i' = \lfloor Q_i \rfloor + 1 = \lceil Q_i \rceil$.