Quiz 9 Solutions

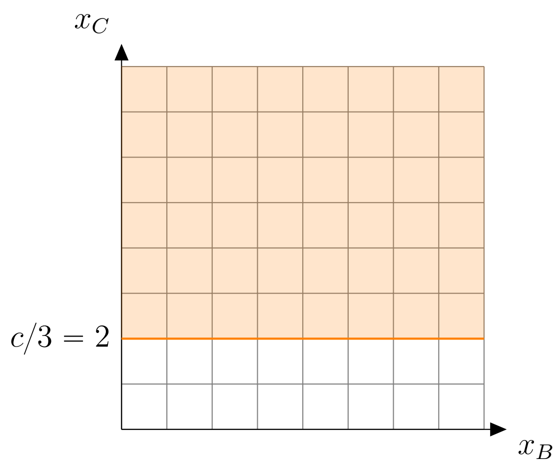
1. **Throughout this question, A and C have bids**

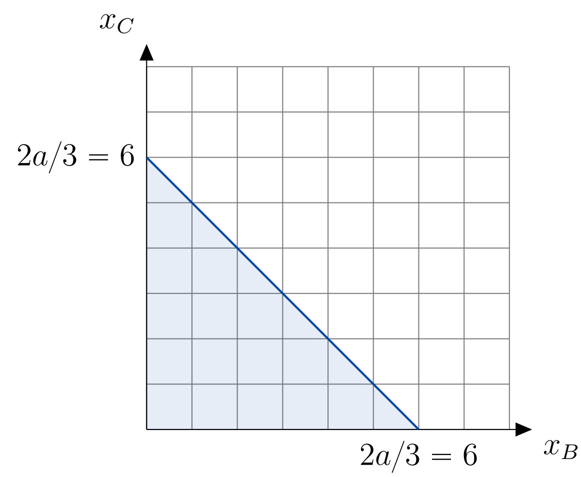
$$a = 9 \qquad c = 6$$

and A is the winning bidder. B's bid will be different depending on certain parts of the question.

(a) In the (x_B, x_C)-plane below on the left, describe all compensation arrangements fair to C.

(b) In the (x_B, x_C)-plane below on the right, describe all compensation arrangements fair to A.





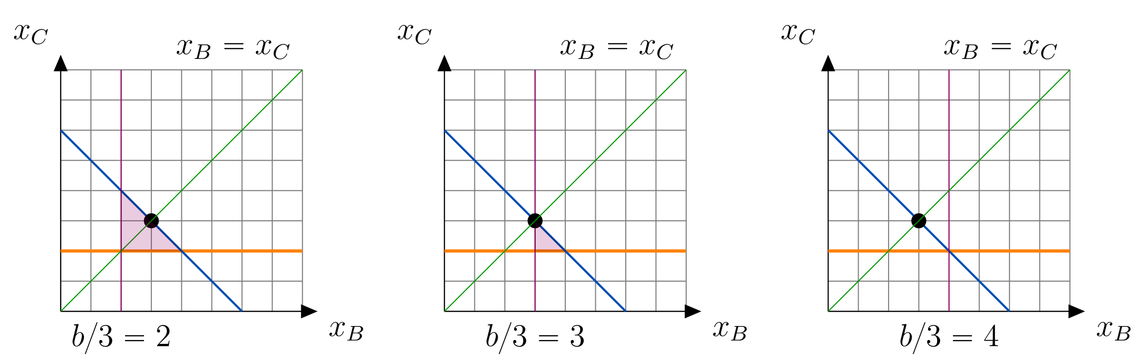
(c) Draw the line representing C getting C's fair share and the line representing A getting A's fair share on the same (x_B, x_C)-plane below. <u>Plot the point $(3, 3)$ in the plane.</u>

i. On this same plane, draw a line representing B getting B's fair share if $b = 6$. Label this line $b = 6$.

ii. On this same plane, draw a line representing B getting B's fair share if $b = 9$. Label this line $b = 9$.

iii. On this same plane, draw a line representing B getting B's fair share if $b = 12$. Label this line $b = 12$.

iv. Graph the line $x_C = x_B$, ie- $y = x$, and label it.



(d) For which values of b is the point $(3,3)$ in the fairness triangle? Circle:

b=6 b=9 $b = 12$

(e) Applying Proposition 13.14 part 3., for which values of b does the point $(3,3)$ correspond to an envy-free compensation arrangement? Circle:

b=6 b=9 $b = 12$

(f) Are there any values of b for which NO envy-free compensation arrangement is possible? If yes, circle them:

$b = 6$ $b = 9$ b=12

(g) True or False:

In this example, an envy-free arrangement is possible if and only if $b \leqslant 9$. T F

(h) **Prove for any bid values**: The winning bidder, A, is a highest bidder if and only if $x_B = x_C = \frac{a}{3}$ is a fair compensation arrangement.

Proof. First, if A is a highest bidder then $a \geqslant b$ and $a \geqslant c$. Then clearly this compensation arrangement is fair to B and C because

$$x_B = \frac{a}{3} \geqslant \frac{b}{3} \qquad x_C = \frac{a}{3} \geqslant \frac{c}{3}$$

This compensation arrangement is also fair to A because A get's exactly A's fair share:

$$x_A = a - x_B - x_C = a - \frac{a}{3} - \frac{a}{3} = \frac{3a - 2a}{3} = \frac{a}{3}$$

To complete the proof, we also need to argue this opposite direction: that $x_B = x_C = \frac{a}{3}$ being fair implies a is a highest bidder. This is clear because everything we said above is an equivalence, so we're done.

□

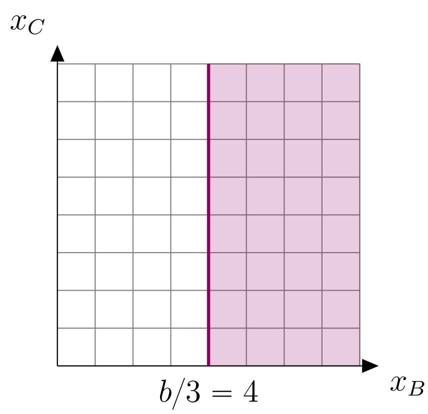
2. **Throughout this question, B and C have bids**

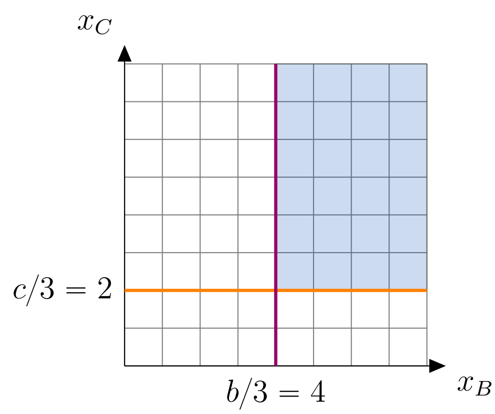
$$b = 12 \qquad c = 6$$

and A is the winning bidder. A's bid will be different depending on certain parts of the question.

(a) In the (x_B, x_C)-plane below on the left, describe all compensation arrangements fair to B.

(b) In the (x_B, x_C)-plane below on the right, describe all compensation arrangements fair to BOTH B and C.





(c) Draw the line representing B getting B's fair share and the line representing C getting C's fair share on the same (x_B, x_C)-plane on the next page.

i. Plot the point which represents both B and C getting exactly their fair shares. What is this point?

$$P = (4, 2)$$

ii. On this same plane, draw a line representing A getting A's fair share if $a = 12$. Label this line $a = 12$.

iii. On this same plane, draw a line representing A getting A's fair share if $a = 9$. Label this line $a = 9$.

iv. On this same plane, draw a line representing A getting A's fair share if $a = 6$. Label this line $a = 6$.

(d) For which values of a was a fair arrangement possible? Circle:

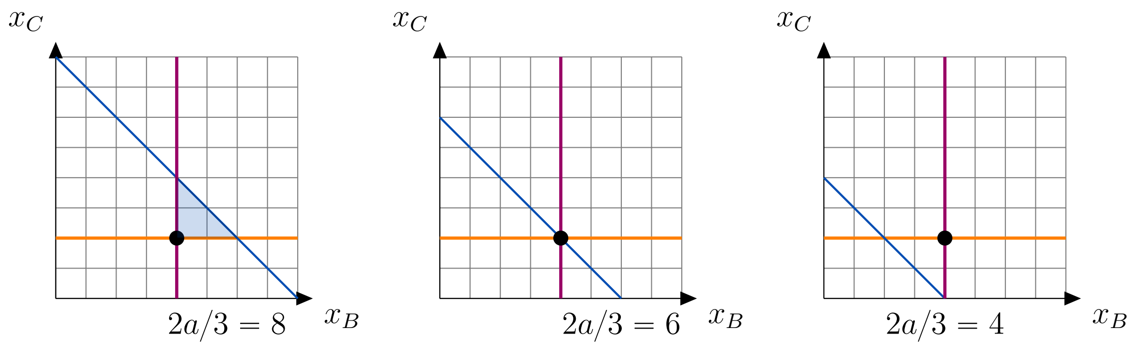
$a = 6$ $\qquad$ $\boxed{a=9}$ $\qquad$ $\boxed{a=12}$

(e) What is the average bid when

$a = 6$? $m = \frac{6+6+12}{3} = 8$

$a = 9$? $m = \frac{6+9+12}{3} = 9$

$a = 12$? $m = \frac{6+12+12}{3} = 10$



(f) Prove: for any real numbers a, b, and c,

$$a = \frac{a+b+c}{3} \iff a = \frac{b+c}{2}$$

Proof.

$$\begin{aligned} a = \frac{a+b+c}{3} &\iff a - \frac{a}{3} = \frac{b+c}{3} \\ &\iff \frac{2a}{3} = \frac{b+c}{3} \\ &\iff a = \frac{b+c}{2} \end{aligned}$$

□

(g) Prove: for any real numbers a, b, and c,
If A, the winning bidder, is an average bidder, then $(b/3, c/3)$ is a point on the line representing A getting A's fair share. Conclude that the fairness triangle is only equal to one point.

Proof. We want to show $(b/3, c/3)$ is a point on the line $\frac{2a}{3} = x_B + x_C$, which represents A getting A's fair share. In other words, we need to show $x_B = b/3$, $x_C = c/3$ is a solution to the equation $\frac{2a}{3} = x_B + x_C$.
By our proof above, we saw that A is an average bidder if and only if $a = \frac{b+c}{2}$. Plugging this in to the left hand side of the equation above, we simplify and get the result:

$$\frac{2}{3} \times a = \frac{2}{3} \times \frac{(b+c)}{2} = \frac{b+c}{3} = \frac{b}{3} + \frac{c}{3}$$

The point $(b/3, c/3)$ is the intersection of the lines representing fairness to B and fairness to C. If the line representing fairness to A hits $(b/3, c/3)$ also, then the point must be the only compensation arrangement fair to all three.

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