Redistricting and the Isoperimetic Problem

Math 19-03 Lecture 23

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Announcements

- Exam: Monday May 4th, 7:00-10:00pm
- Office hours next week: Monday 7:30-9:00pm, Wednesday 10:30am-12:00pm
- Bonus rounds:
  1. Thursday April 30th, TBD, 10:30am-12:00pm (check course webpage for details)
  2. Sunday May 3rd, Robinson 152, 7:00-8:30pm

Today

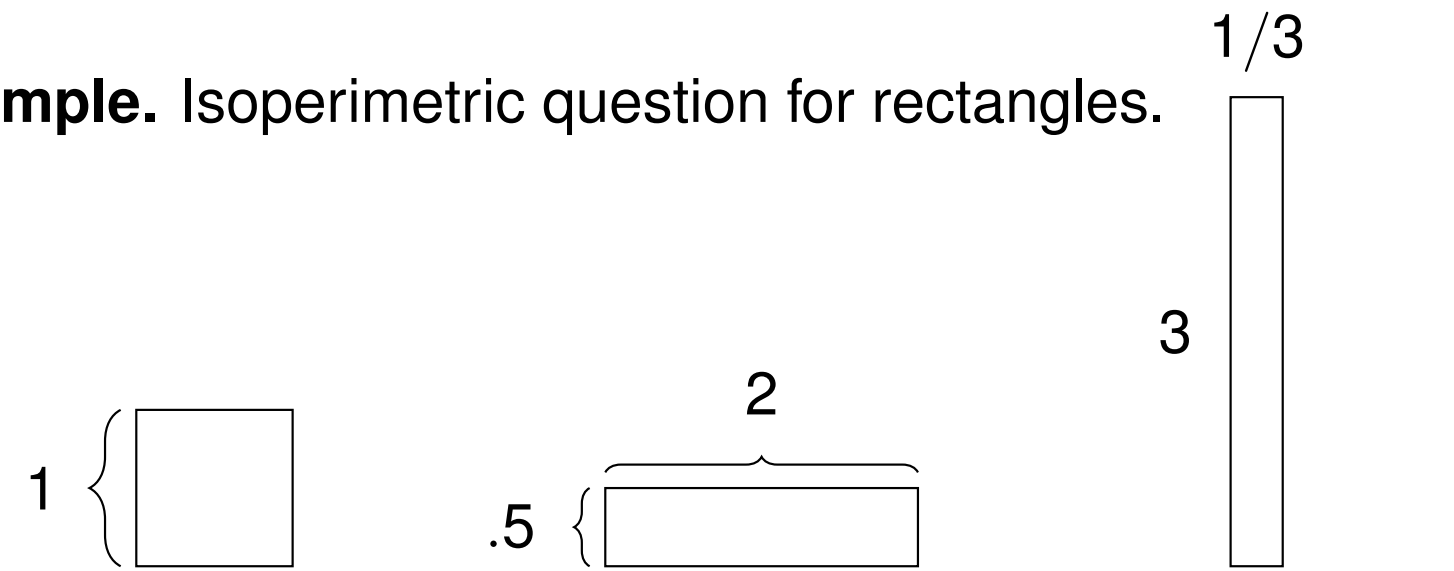
The story of Dido ($\sim$ 825 BCE)

Dido's question: How much land can be bound by a bull's hide?

Source: Ashbaugh & Benguria

The Isoperimetric Problem: how to maximize area with a fixed perimeter?
Rephrased: how to minimize perimeter with a fixed area?

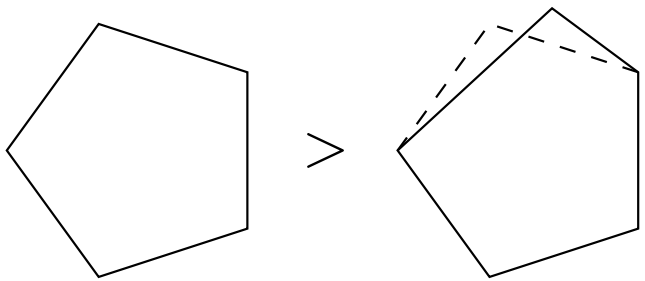
Example. Isoperimetric question for rectangles.



$P = 4,\ A = 1$ $P = 5,\ A = 1$ $P = 6\frac{2}{3},\ A = 1$

Fact. The square solves the isoperimetric problem for rectangles: minimum perimeter per unit area.

Theorem. (Zenodorus, ∼200 BCE) A regular polygon with n sides solves the isoperimetric problem for n-sided polygons.

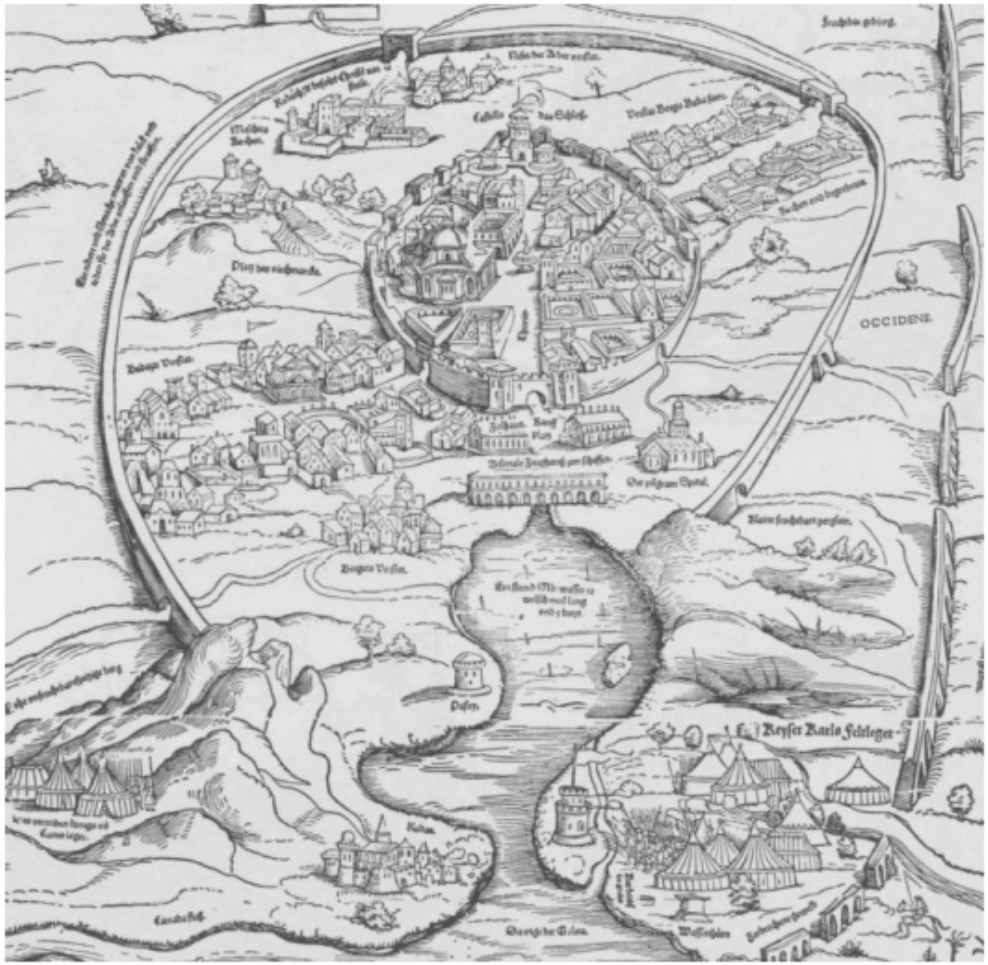


Is there a shape which maximizes an area A for a given perimeter L?

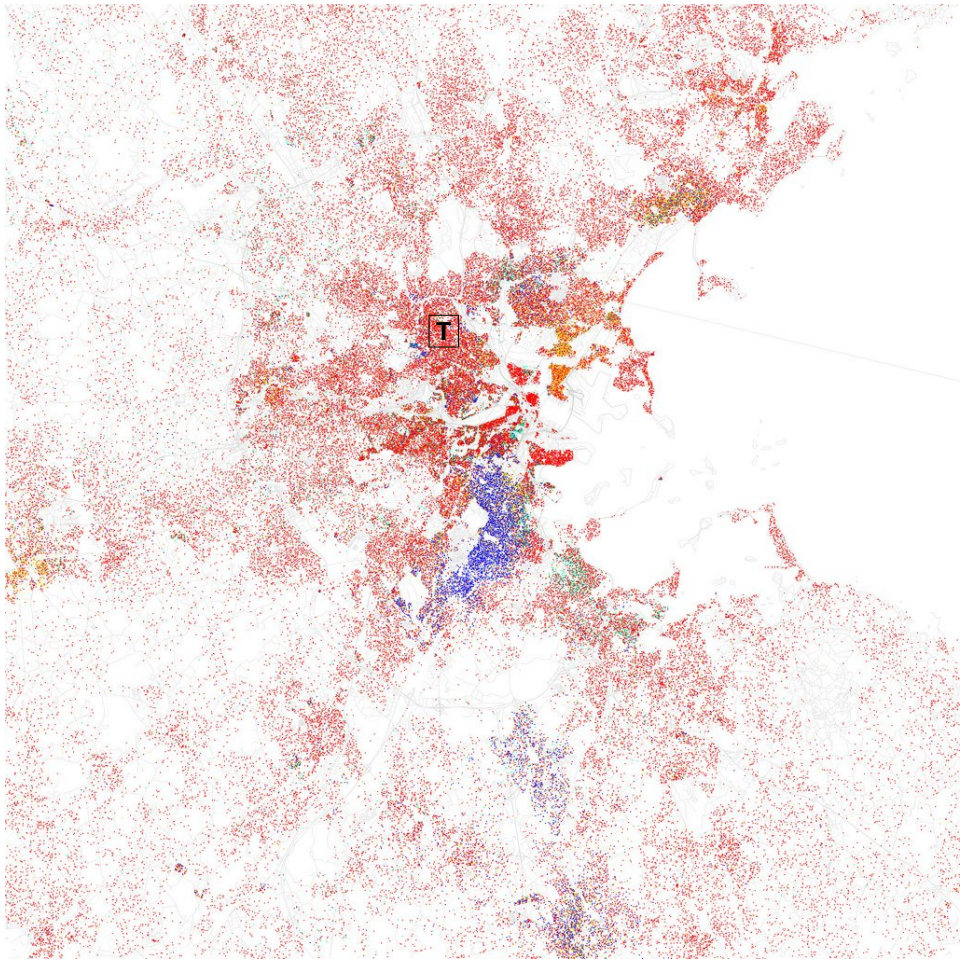
The Isoperimetric Theorem (Steiner, 1842).

Yes! It's the circle of circumference L

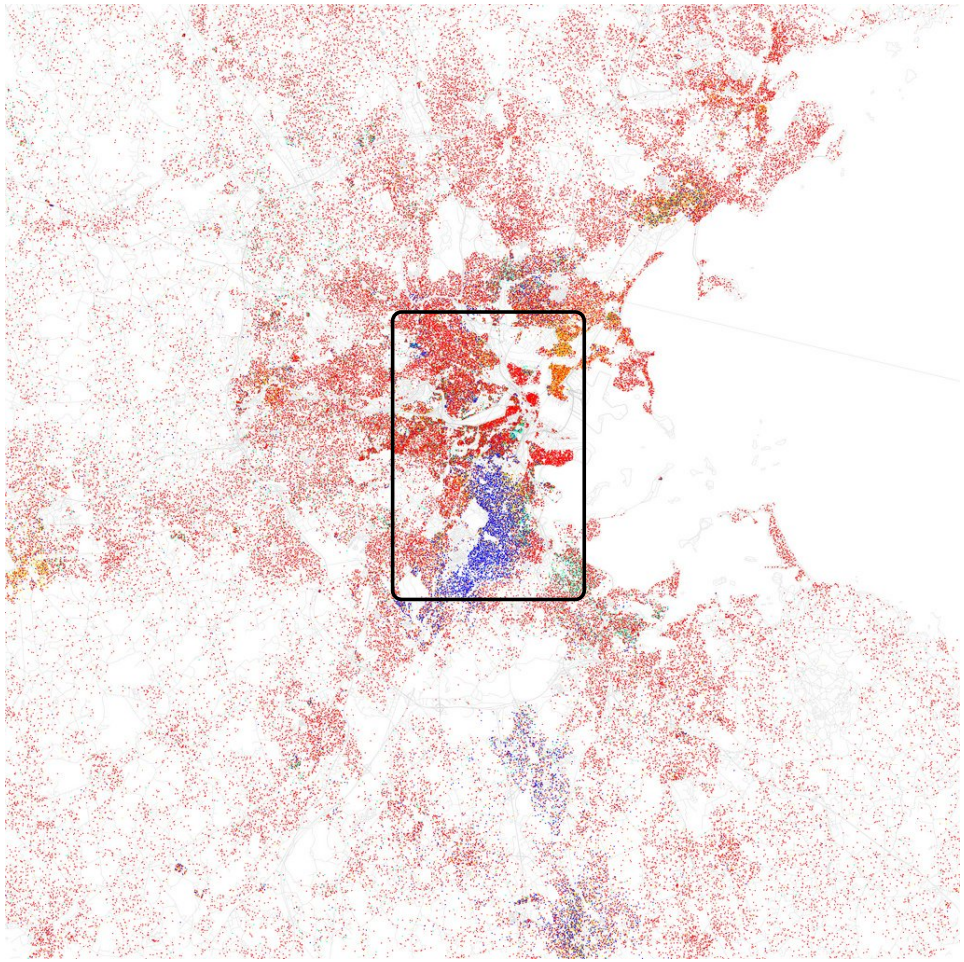
The Isoperimetric Inequality. $L^2 - 4\pi A \geq 0$, with equality only for the circle.



Detail from a sketch made in commemoration of Carlos Quintos' campaign on the doubled-walled city of Tunis, clearly satisfying the isoperimetric property of the circle. (31 August 1535). Source: Ashbaugh & Benguria



City of Boston Demographic Information: The red dots show white people, blue is black, orange is Hispanic, green is Asian, and yellow is other, according to maps of 2010 Census data by Eric Fischer.
Source: Business Insider



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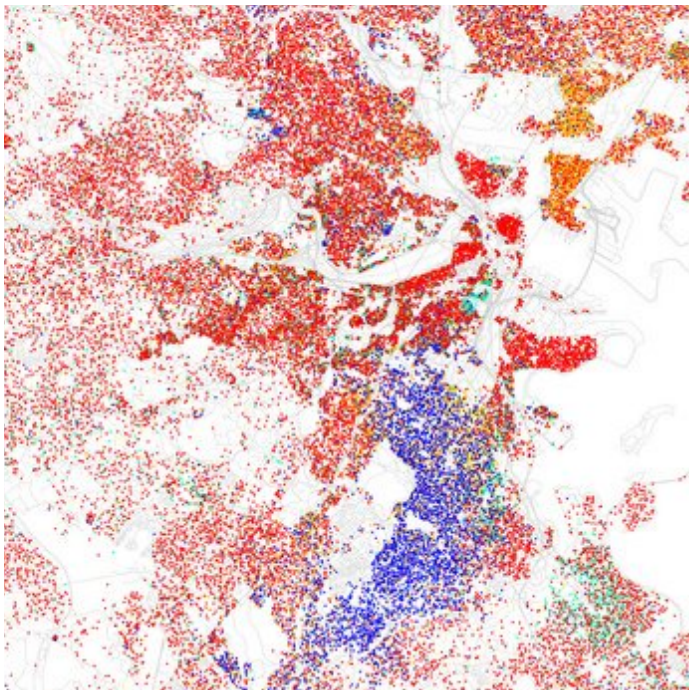
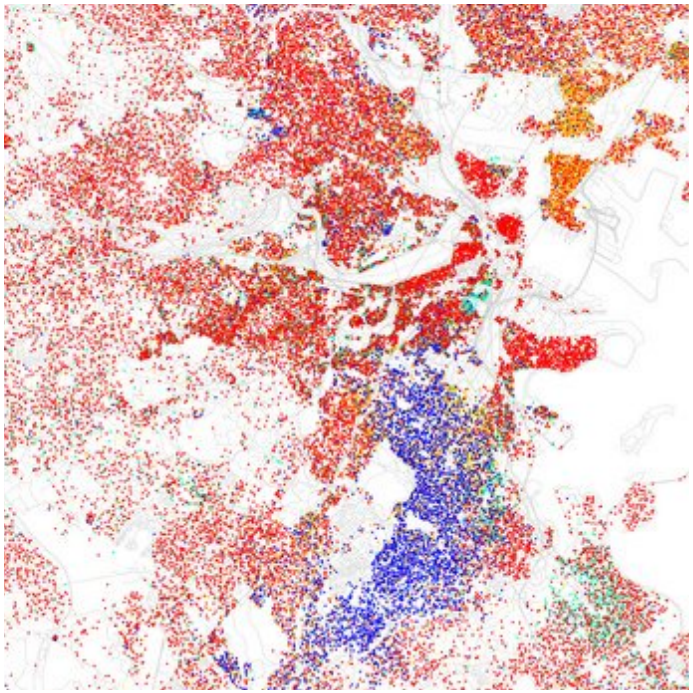
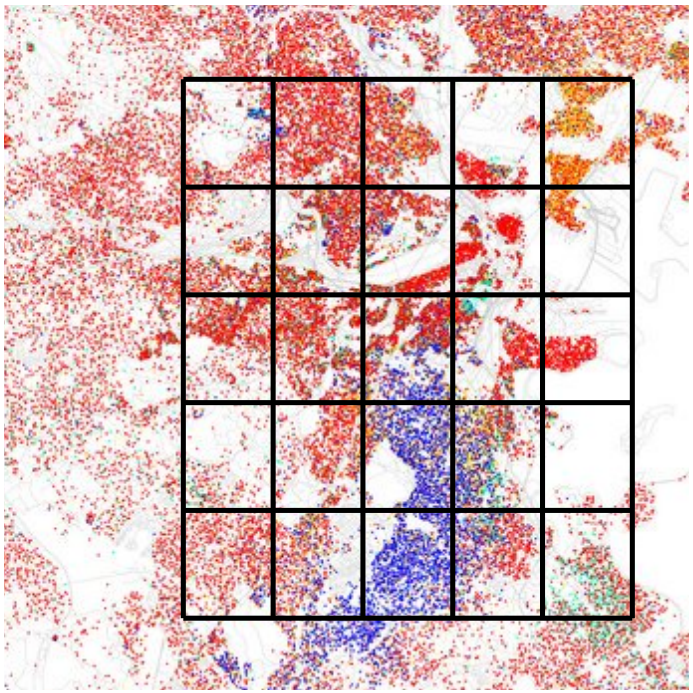
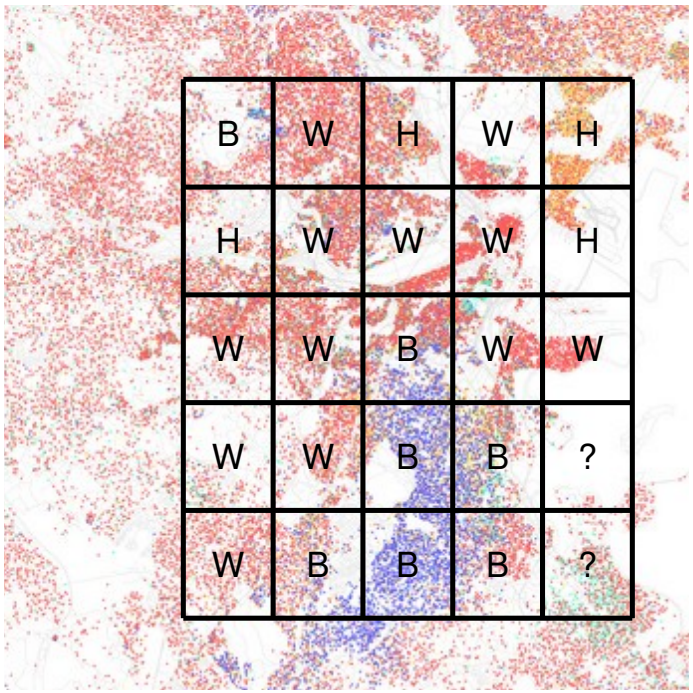
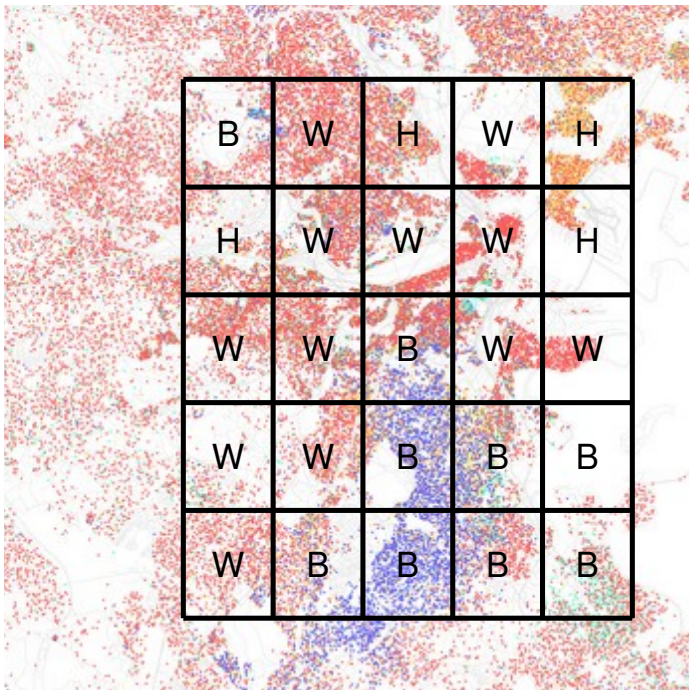




The original Gerry-Mander of the Jefferson Party in 1812.
Source: SIAM news



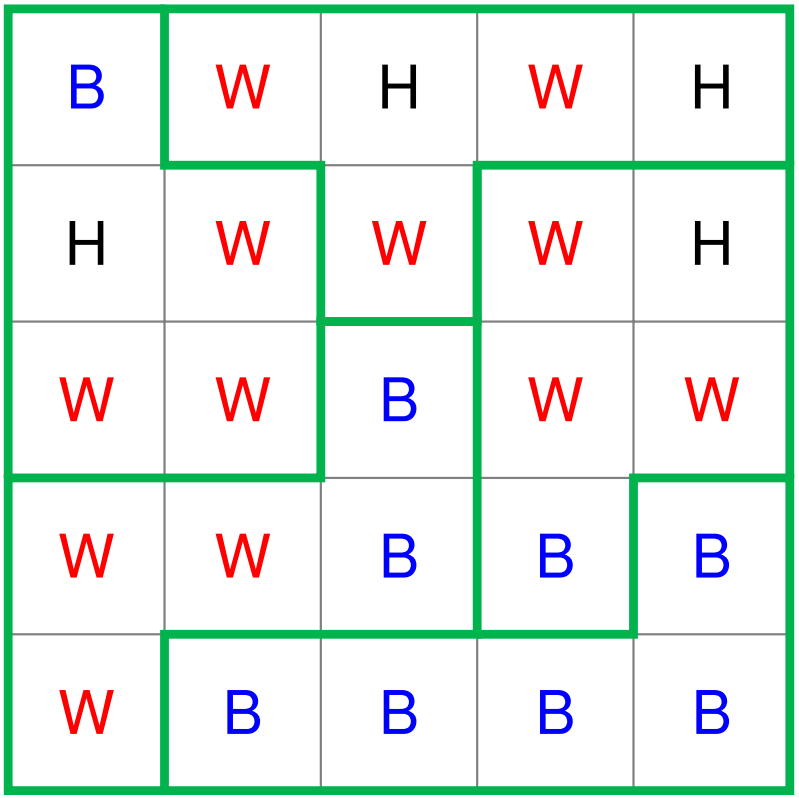

B W H W H
H W W W H
W W B W W
W W B B ?
W B B B ?


B W H W H
H W W W H
W W B W W
W W B B B
W B B B B

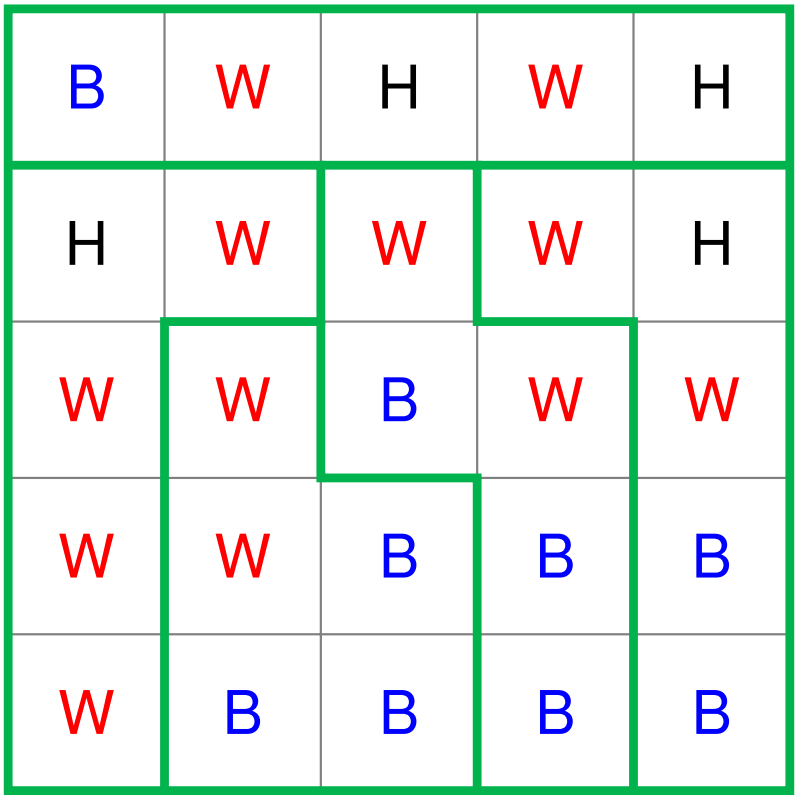
25 neighborhoods, 5 districts equal size, 1 rep each district.

B	W	H	W	H
H	W	W	W	H
W	W	B	W	W
W	W	B	B	B
W	B	B	B	B

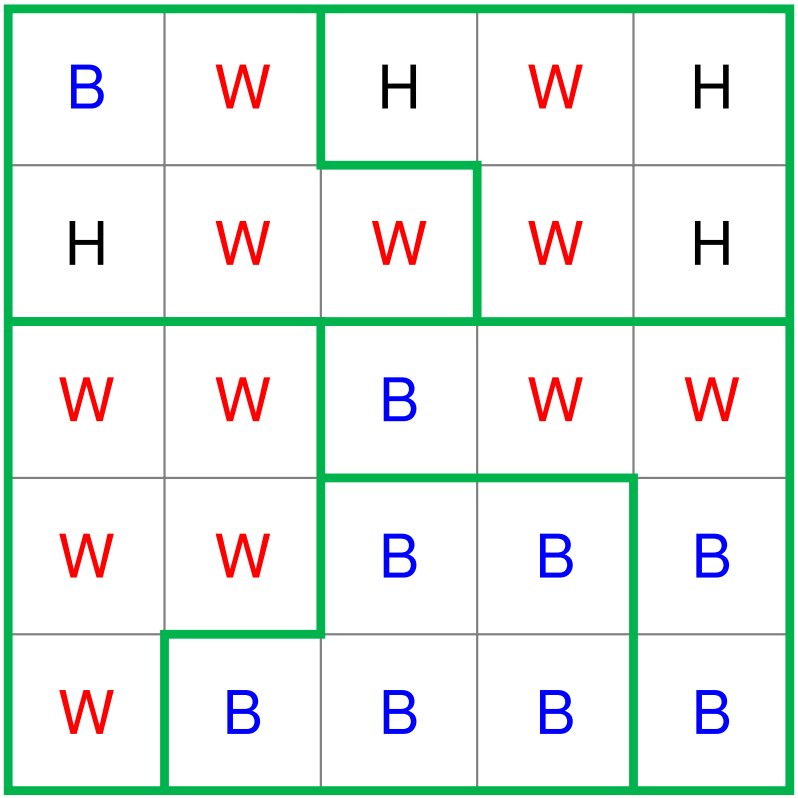
- # W = 12 $\rightarrow$ 2.4 reps
- # B = 9 $\rightarrow$ 1.8 reps
- # H = 4 $\rightarrow$ .8 reps



- W: 4 reps
- B: 1 rep
- H: 0 reps



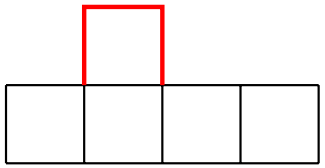
- W: 1 full, 2 half $\rightarrow$ 2 reps
- B: 2 full, 1 half $\rightarrow$ 2.5 reps
- H: 0 full, 1 half $\rightarrow$ .5 reps

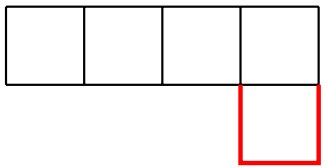


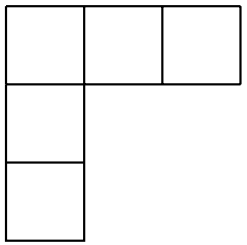
- W: 2 reps
- B: 2 reps
- H: 1 rep

Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?

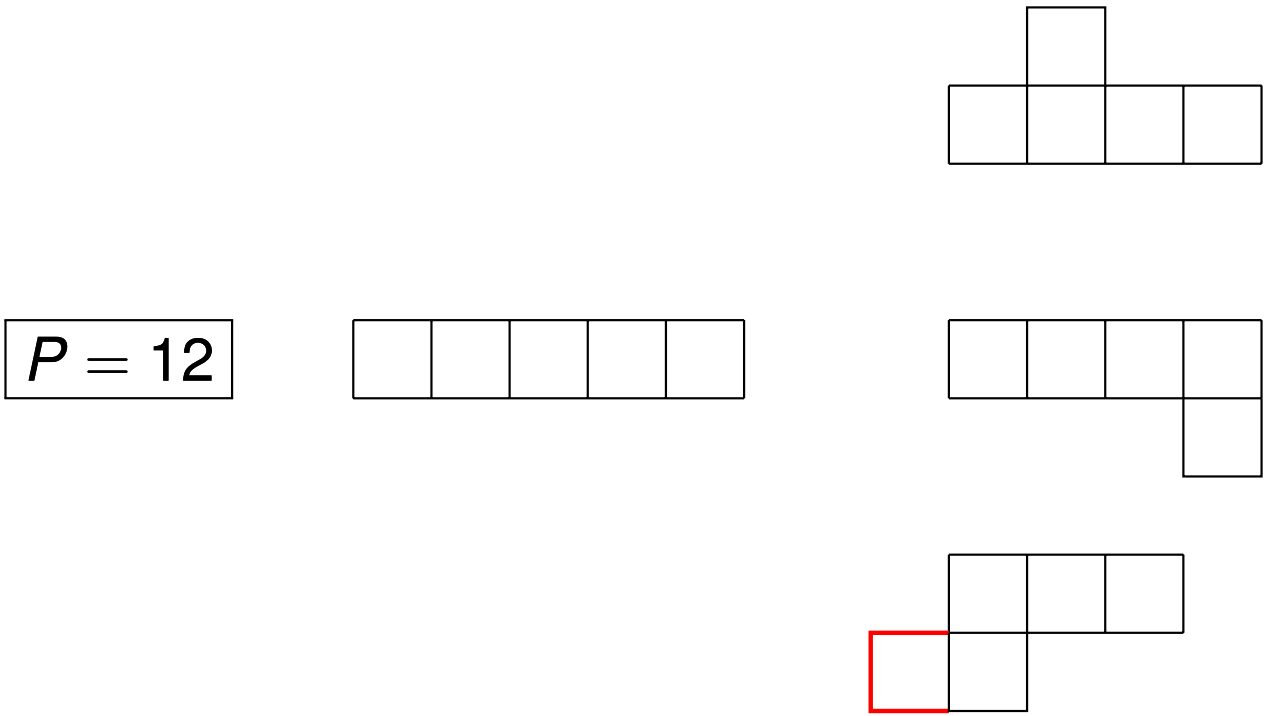
$P = 12$







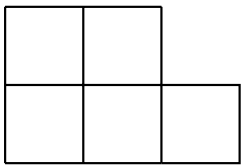
Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?



Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?

$P = 12$

"spread"



$P = 10$

"compact"

Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?

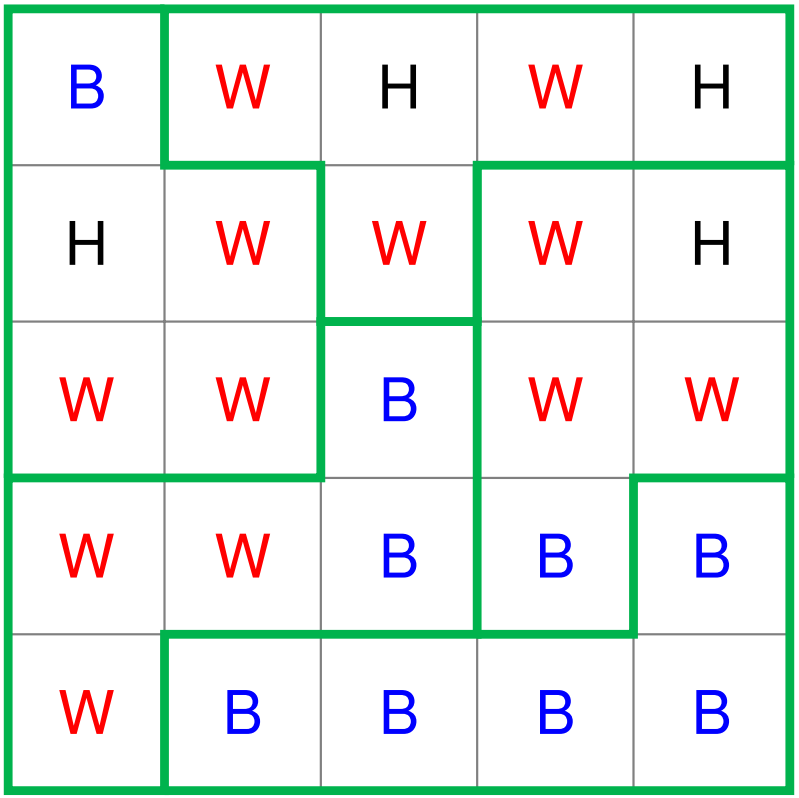
$P = 12$

"spread"

$P = 10$

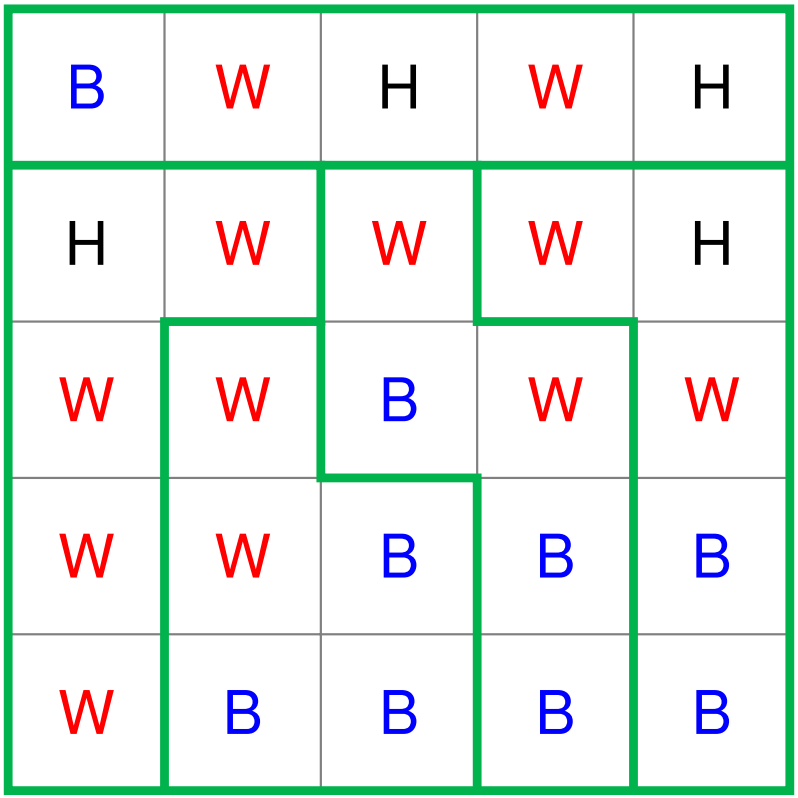
"compact"

An isoperimetrically efficient districting scheme will have more compact type districts and fewer spread type districts.



- W: 4 reps
- B: 1 rep
- H: 0 reps

spread = 3
compact = 2
Inefficient



- W: 1 full, 2 half $\rightarrow$ 2 reps
- B: 2 full, 1 half $\rightarrow$ 2.5 reps
- H: 0 full, 1 half $\rightarrow$ .5 reps

spread = 4
compact = 1
Inefficient

B	W	H	W	H
H	W	W	W	H
W	W	B	W	W
W	W	B	B	B
W	B	B	B	B

- W: 2 reps
- B: 2 reps
- H: 1 rep

spread = 1
compact = 4
Efficient
Optimal !!!

Isoperimetric Problems Today

- Computer scientists study the isoperimetric problem for districting in the United States: Washington Post
- My officemate digs it, for his research about math: Exploring Isoperimetric Inequalities in Heisenburg space.

Review time!

Resources on course webpage