Fairness triangle for compensation arrangements

1. Set A's bid to 60, B's bid to 45, and C's bid to 30. Activate the Equitable arrangement by clicking the box next to it. Do the same for the Equal compensation arrangements.

   (a) Slide B's bid to the left until the equitable arrangement is on the green line, $x_B = x_C$. What must be true for B's and C's bids for this to happen?

   (b) Slide A's bid to the far right and far left. Geometrically, what is happening to the equitable arrangement?

   (c) Which equitable arrangements in 1(b) are envy-free?

   (d) Prove: Assume A is a highest bidder. Then the equitable arrangement is envy-free if and only if B's bid $=$ C's bid.

   Proof. Since A is a highest bidder, the envy-free arrangements are precisely those for which $x_B = x_C$ and the arrangement is fair. Any highest bidder is an average bidder, so the equitable arrangement will be fair. So we just need to show that the compensation amounts for the equitable arrangement are equal if and only if B's bid $=$ C's bid:

$$x_B = qb = qc = x_C$$
$$\iff \not{q}b = \not{q}c \iff b = c \qquad \text{because } q \neq 0 \text{ for } a, b, c > 0$$

□

2. In this question, we will explore what the compensation arrangements look like when the winning bidder A is an average bidder.

   (a) Recall from Quiz 9 that we proved for <u>any</u> real numbers a, b, and c:

   $$a = \frac{a+b+c}{3} \iff a = \frac{b+c}{2}$$

   Now, In the applet, set B's bid to 30 and C's bid to 50. What must A's bid be for A to be exactly the average bidder?

   A's bid = ________

   (b) Slide A's bid to that value. What does the fairness triangle look like? Give the explicit value(s) for all point(s) in the fairness triangle.

   (c) Are there any envy-free compensation arrangements? Explain geometrically.

   (d) Prove for any bid values: If A is exactly an average bidder, then every fair compensation arrangement is equitable.

   Proof. Since A is average, there is only one fair compensation arrangement: $x_B = \frac{b}{3}$ and $x_C = \frac{c}{3}$. That's because if payouts were any larger, then

   $$x_A = a - x_B - x_C < \frac{a+b+c}{3} - \frac{b+c}{3} < \frac{a}{3}$$

   By the proposition in the book, the equitable arrangement is fair when A is an average bidder. Since there is only one fair arrangement, it must be the equitable one.

   □

3. Set A's bid to 60, B's bid to 40 and C's bid to 30. Move the dynamic point to $\{20, 20\}$. Activate the Equal Compensation Amounts line.

   (a) The point $\{20, 20\}$ is the intersection of two lines on the graph. Which two lines? Circle two:

   $$x_A = \frac{a}{3} \qquad x_B = \frac{b}{3} \qquad x_C = \frac{c}{3} \qquad x_B = x_C$$

   (b) Slide B's bid to the right. For what values of B's bid is the compensation arrangement $x_B = x_C = 20$ fair?

   (c) There is a value for B's bid such that only ONE envy-free arrangement is possible. What is that value?

   B's bid $=$ ________

   (d) For what values of B's bid is an envy-free arrangement possible?

   (e) Prove for any bid values: The winning bidder, A, is a highest bidder if and only if $x_B = x_C = \frac{a}{3}$ is a fair compensation arrangement.

   Proof. First, the arrangement is fair to B and C if and only if A is a highest bidder:

   $$\begin{aligned} A \text{ is a highest bidder} &\iff a \geqslant b \text{ and } a \geqslant c \\ &\iff \frac{a}{3} \geqslant \frac{b}{3} \text{ and } \frac{a}{3} \geqslant \frac{c}{3} \\ &\iff x_B \geqslant \frac{b}{3} \text{ and } x_C \geqslant \frac{c}{3} \end{aligned}$$

   Also, this arrangement is always fair to A because

   $$x_A = a - x_B - x_C = a - \frac{a}{3} - \frac{a}{3} = \frac{a}{3}$$

   □

4. Set A's bid to 60. Set both B and C's bids to 30.
   (a) Compute q for these bid values. Your answer should agree with the applet.
   (b) Slide C's bid to the right by 5. Then decrease B's bid one by one (use plus and minus buttons). How much do you have to decrease B's bid for q to equal 1/2 again?
   (c) Reset B's bid to 30. Keep C's bid at 35. Increase A's bid one by one until $q = 1/2$. How much do you have to increase A's bid to get $q = 1/2$?
   (d) Try other combinations that give $q = 1/2$, by first sliding one bid and then correcting another. What appears to be the relationship between the bids for A, B, and C? Write your guess as an equation in a, b, and c.
   (e) Using algebra, prove that your guess from 3(e) is true. In other words, prove:

$$q = \frac{1}{2} \iff \underline{\text{Your guess here:} \qquad\qquad\qquad\qquad}$$

Proof. The guess should be $a = b + c$.

$$\begin{aligned} q = \frac{1}{2} &\iff \frac{a}{a+b+c} = \frac{1}{2} \\ &\iff 2a = a + b + c \\ &\iff a = b + c \end{aligned}$$

□

Extra credit:

(f) Find bid values with $q = 1/4$, $q = 1/3$, and $q = 3/5$:

$q = \frac{1}{4}$: $a =$ ________ $b =$ ________ $c =$ ________

$q = \frac{1}{3}$: $a =$ ________ $b =$ ________ $c =$ ________

$q = \frac{3}{5}$: $a =$ ________ $b =$ ________ $c =$ ________

(g) Activate the equitable arrangement. For each of the bid combinations you proposed in 3(f), adjust the bid values in the applet. Then answer the following:

i. For which values of q is the arrangement fair?

ii. Using your bid values proposed for each q in the last question, compute $a - b - c$. Compare when q is small to when q is big. Compare also to the q from 4(a)-(e).

iii. How does the fairness triangle compare for bid values with large q to bid values with small q?

iv. Compute the payouts for the three different equitable arrangements you came up with. What do they look like when a is large? Explain conceptually why this makes sense.