

# Compensation Handout

Three math students (named A, B and C) share a book, and when class is over, have to decide who gets it. We'll look at 5 different possible compensation arrangements, and then compare them.

Suppose these are the values each student has for the book:

a = \$ 18

b = \$24

c = \$30

1) Suppose C gets the book and pays A and B each \$2. What is C's payout?

$$X_C = c - X_A - X_B = 30 - 2 - 2 = 26$$

fair? **NO**:  $2 \neq \frac{18}{3} = 6$   
not fair to A  
(or B)

envy-free? not fair  
 $\Rightarrow$  **not envy free!**

equitable? **NO**

Pareto-optimal? **Yes**: C is a highest bidder (Prop 14.2)

Prop 14.9

$c = w = 30 > m$  so the only equitable arrangement must be fair. Since this arrangement is not fair, it cannot be equitable (in this case!)

2) Suppose C gets the book and pays A and B each \$8. What is C's payout?

$$X_C = 30 - 16 = 14$$

fair? **Yes!**

envy-free? A and B do not envy each other, and C does not envy them (C thinks C gets  $X_C > 8$ )

$$X_B = 8 > 8 = b/3 \quad X_C = 14 > 10 = c/3$$

A thinks C gets  $18 - 16 = 2$  no envy  
B thinks C gets  $24 - 16 = 8$  no envy

$$X_A = 8 > 6 = a/3$$

equitable?

Pareto-optimal?

**no**  $X_C \neq X_A = \frac{4}{9}$   
 $\frac{7}{15} = \frac{c}{a}$

**yes** again by prop 14.2

3) For an equitable and fair arrangement, who can get the book?

B and C are both average so we could give it to either and find an equitable arrangement by prop 14.9 (at least)

Suppose B gets the book. Find an equitable, and fair arrangement.

$$q = \frac{w}{S} = \frac{24}{18+24+30} = \frac{24}{72} = \frac{1}{3}$$

$$X_A = qa = \frac{a}{3} = \boxed{6}$$

$$X_B = qb = \boxed{8}$$

$$X_C = qc = \boxed{10}$$

check:  $6 + 8 + 10 = 24 = w$

is this envy-free? **NO**

Pareto-optimal? **NO**, B is not a highest bidder (Prop 14.2)

↙ C winning bidder ↘

$$q = \frac{c}{8} = \frac{30}{12} = \frac{5}{2}$$

4) Find a **Pareto-optimal, fair, equitable** arrangement. Who gets the book?

paychecks  $\left\{ \begin{array}{l} X_A = qa = \frac{5}{2} \cdot 18 = 15 \\ X_B = qb = \frac{5}{2} \cdot 24 = 30 \end{array} \right.$

value of book to C  $- X_C = qc = \frac{5}{2} \cdot 30 = 75$

check  $X_A + X_B + X_C = W$   
 $\frac{15}{2} + \frac{30}{2} + \frac{75}{2} = \frac{120}{2} = 60 \checkmark$

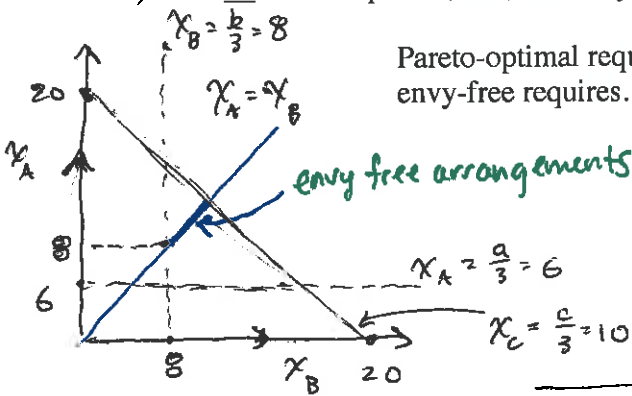
is this envy free?

**NO** A and B get different paychecks.

is this an objective improvement over arrangement 3?

**Yes** - literally everyone got a bigger payout

5) Find all Pareto-optimal, fair, and envy-free arrangements:



Pareto-optimal requires... C winning bidder  
 envy-free requires...  $X_A = X_B$  and  $\frac{b}{3} = 8 \leq X_B \leq \frac{c}{3} = 10$

Prop 13.10: envy-free  $\Leftrightarrow$  all compensation amounts equal and the arrangement is fair  
 these are described by

$$X_B = X_A, \quad 8 \leq X_B \leq 10, \quad X_C = 30 - X_A - X_B$$

6) Looking at all the arrangements, there are two categories of better arrangements:

What arrangements were fair, Pareto-optimal, and equitable?

# 4

What arrangements were fair, Pareto-optimal, and envy-free?

# 2, everything in # 5 (note that the payout in # 2 represents the intersection of  $X_A = X_B$  with  $X_B = 8$ )

Is a fair, Pareto-optimal, equitable, envy-free arrangement possible for this book?

Nope. the fair, Pareto-optimal, equitable arrangement in 4 is uniquely determined, meaning there is only one. Since it was not envy-free, we can never have all the things (tragedy).