Homework 5: Chapter 3

Reading: Chapter 3 pgs 17-20, Appendix A pgs 195-198
Exercises:

1. From the book: 3.1, 3.2, and 3.4 (Book solutions.)

2. Use the following example to show that sequential comparison fails the unanimity criterion, with candidates ordered alphabetically:

1	1	1
A	C	B
B	A	D
D	B	C
C	D	A

Solution. With the candidates ordered alphabetically, we first pit A against B and determine that A wins. Then A faces C and loses to C. Lastly, C faces D, who wins. With this ordering, D is the only winner by sequential comparison.

This example shows that sequential comparison fails the unanimity criterion because D was unanimously ranked behind B, but D still won... somehow. Rude.

□

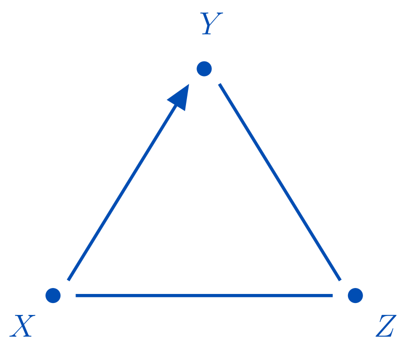
3. Our solution to Exercise (2.5) from class happens to also show that pairwise comparison admits losing spoilers. I'll recall the example here:

1	1
X	Z
Y	X
Z	Y

Show that there is a losing spoiler for ALL POSSIBLE examples satisfying the conditions in Exercise (2.5), as long as $n = 3$. (Hint below)

Bonus round: (optional) construct an example with $n = 4$ which satisfies the conditions of Exercise #2.5, but doesn't have any spoilers.

Solution. Let X, Y, Z be the $n = 3$ candidates. The conditions in Exercise 2.5 require that X beats Y in a head-to-head battle, Z and X tie, and Z and Y tie. Therefore, the Comparison graph will always look like this:



Since X gets 1.5 pairwise comparison points, Z gets 1 pairwise comparison point, and Y gets .5 pairwise comparison points in this pairwise comparison tournament, we have $W = \{X\}$ for pairwise comparison.

We now want to show that there is always a spoiler for pairwise comparison in such an election. There cannot be any winning spoilers by definition because there is only one winner. So we check if Y or Z are losing spoilers.

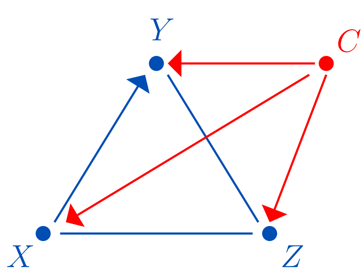
Suppose Y was disqualified. We deduced in the Hint that the new Comparison graph looks the same, just with all arrows to Y forgotten:

In this pairwise comparison tournament, $W' = \{X, Z\}$ because X and Z tie. Since $W' \neq W$, we conclude that Y is a losing spoiler.

Technically we are done, but we might still wonder... Could Z be a losing spoiler?

$\square$

Bonus round. We will show the assumption that $n \leqslant 3$[1] was a necessary condition, meaning if $n \geqslant 4$, the claim is false. Here is the comparison graph for an example when $n = 4$ satisfying the conditions of Exercise (2.5), but there are NO losing spoilers:



[1]you can check that the claim is true for $n = 1, 2$.

Notice first that X, Y, and Z still satisfy the conditions of Exercise (2.5). However, the new fourth candidate C is a Condorcet candidate by design. In Exercise 5 below, we'll see that if there is a Condorcet candidate, there are no spoilers for pairwise comparison, and we have the conclusion for the question.

You should check, by the way, that this comparison graph does indeed come from a preference schedule - in other words, find a preference schedule which has this comparison graph. Not every graph comes from a preference schedule!

□

4. Prove that if there is a Majority candidate, there will **not** be any spoilers for **plurality**.

   Proof.

   - Suppose M is a Majority candidate. Then M wins alone by plurality because plurality is majority fair.
   - If there is exactly one candidate in W, there cannot be a winning spoiler. In math speak, $W = \{M\}$.
   - Let's check for losing spoilers. Suppose any other candidate Y is eliminated. Even if any other candidate leaves, M still has the majority of the vote, and wins alone again by plurality. The new winner set is $W' = \{M\}$.
   - Since the winners do not change when Y is disqualified, we conclude Y is not a losing spoiler.
   - Candidate Y was arbitrary, so we can say there are NO losing spoilers for plurality under the condition that there is a Majority Candidate.

   □

5. Prove that if there is a Condorcet candidate, then there will **not** be any spoilers for **pairwise comparison**. Complete Exercise (3.4) to notice that Condorcet candidates can be spoilers for **plurality**.

   Proof. Replace the words "majority" with "Condorcet" and "plurality" with "pairwise comparison" in the above proof, and everything is still true.

   Since plurality is NOT Condorcet fair, it should not be surprising that we are able to find Condorcet candidates who are losing spoilers for plurality. You'll see one in Exercise 3.4. □

Hint to Exercise 2: Draw the comparison graph for the example in class. Can you draw it for any example with $n = 3$ under the conditions of Exercise 2.5?

Now, consider an election with any number of n candidates. Consider the comparison graph. If you remove one candidate, how does the new comparison graph change?