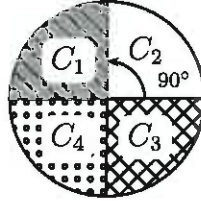
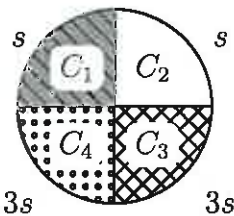


1. Three cousins Edward ( $E$ ), Sam ( $S$ ) and Rebecca ( $R$ ) have been told they inherited the circular plot of land which is made up of 4 components,  $C_1, C_2, C_3$  and  $C_4$ , each composing  $1/4$  of the plot.

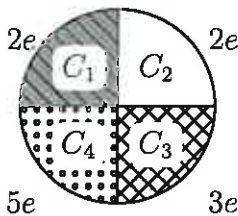


$E, S$  and  $R$  decide they will divide up the land between the three of them. Their preferences are as follows:

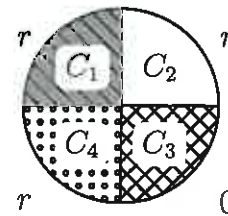
$S$ 's preferences:



$E$ 's preferences:



$R$ 's preferences:



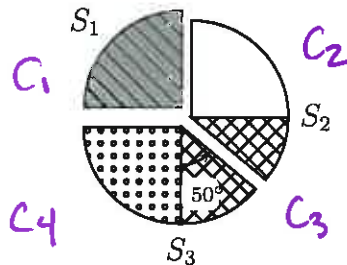
Their valuations of each component are given the table below.

	$C_1$	$C_2$	$C_3$	$C_4$
$E$	$2/12$	$2/12$	$3/12$	$5/12$
$S$	$1/8$	$1/8$	$3/8$	$3/8$
$R$	$1/3$	$1/3$	$0$	$1/3$

*use this as an opportunity to practice computing values!*

- (a) Suppose first that they decide to use the **lone divider** method to divide up the land. They decide that
- $R$  is the divider, and
  - $E$  and  $S$  are the choosers.

Suppose  $R$  cuts as follows:



i. Fill in the blanks with the fraction of each component that occurs in each slice (this is just a volume measurement):

$$\begin{array}{l}
 S_1 = \frac{1}{4} C_1 + \frac{0}{4} C_2 + \frac{0}{4} C_3 + \frac{0}{4} C_4 \\
 S_2 = \frac{0}{4} C_1 + \frac{1}{4} C_2 + \frac{1/2}{4} C_3 + \frac{0}{4} C_4 \\
 S_3 = \frac{0}{4} C_1 + \frac{0}{4} C_2 + \frac{1/2}{4} C_3 + \frac{1}{4} C_4
 \end{array}$$

	$S_1$	$S_2$	$S_3$
$C_1$	1	0	0
$C_2$	0	1	0
$C_3$	0	1/2	1/2
$C_4$	0	0	1

ii. Fill in the valuations of each slice below using the denominators given:

	$C_1$	$C_2$	$C_3$	$C_4$		$S_1$	$S_2$	$S_3$	Bid list
E	$1/6$	$1/6$	$1/4$	$5/12$	E	$1/6$	$7/24$	$13/24$	$S_3$
S	$1/8$	$1/8$	$3/8$	$3/8$	S	$1/8$	$5/16$	$9/16$	$S_3$
R	$1/3$	$1/3$	0	$1/3$	R	$1/3$	$1/3$	$1/3$	$S_1, S_2, S_3$

iii. List all slices each of  $S$  and  $E$  think is worth at least  $1/3$  of the land in his/her own eyes.

$$* \frac{1}{3} = \frac{5}{15} > \frac{5}{16}$$

$S$ :  $S_3$   
 $E$ :  $S_3$

iv. List all divisions that could result from using the lone divider method where  $R$  cuts as shown above and  $S$  and  $E$  are the choosers.

Could give  $S_1$  to  $R$ , play 1 cut you choose btwn  $E$  and  $S$  over  $S_2 + S_3$

OR give  $S_2$  to  $R$ , play 1 cut you choose btwn  $E$  and  $S$  over  $S_1 + S_3$ .

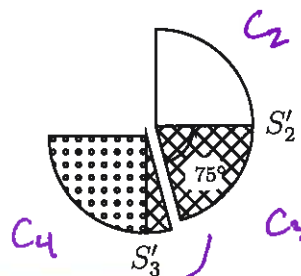
There are a lot of divisions that could result... Bonus points if you go construct one that is different from what follows.

v. Suppose the outcome is that  $R$  gets  $S_1$  and I cut, you choose is used to divide the combined slice  $S_2 + S_3$  between  $S$  and  $E$ . They flip a coin for see who divides and it turns out that

- $S$  is the divider, and
- $E$  is the chooser.

$S$  decides to cut as follows:

$$\frac{75}{90} = \frac{5}{6}$$



$$S_2' = 1C_2 + \frac{5}{6}C_3$$

$$S_3' = \frac{1}{6}C_3 + 1C_4$$

A. What fraction of the whole cake is the mini cake  $S_2 + S_3$  in each of  $S$  and  $E$ 's eyes?

$E$ :  $\frac{20}{24} = \frac{5}{6}$

$E$ : \_\_\_\_\_

$S$ :  $\frac{7}{8}$

$S_2'$     $S_3'$   
 $C_1$    0   0  
 $C_2$    1   0  
 $C_3$     $5/6$     $1/6$   
 $C_4$    0   1

B. Fill in the blanks with the fraction of each component that occurs in each slice (this is just a volume measurement):

$$S_2' = \underline{0} C_1 + \underline{1} C_2 + \underline{5/6} C_3 + \underline{0} C_4$$

$$S_3' = \underline{0} C_1 + \underline{0} C_2 + \underline{1/6} C_3 + \underline{1} C_4$$

C. Fill in the valuations of each slice as ~~fractions of the mini-cake,  $S_2 + S_3$~~  below using the denominators given:

	$S_2'$	$S_3'$
E	$3/8$ $\times_{20}$	$11/24$ $\times_{20}$
S	$7/16$ $\times_{1/2}$	$7/16$ $\times_{1/2}$

Fractions of the whole cake

D. List the division that could result from I cut, you choose in this case.

$$S : \underline{S_2'}$$

$$E : \underline{S_3'}$$

E. Given your result above, list the shares resulting from using the lone divider method (so these are the fraction of the cake each person thinks they receive in their own eyes as a fraction of the whole cake,  $C_1 + C_2 + C_3 + C_4$ ).

$$E's \text{ share} = \underline{11/24}$$

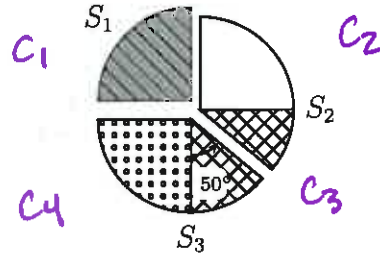
$$S's \text{ share} = \underline{7/16}$$

$$R's \text{ share} = \underline{1/3}$$

(b) Rebecca recently learned about the method of **Selfridge and Conway** and that it is an envy-free method, which lone divider is not. Therefore, the three now decide to divide the case using the method of **Selfridge and Conway** where

- $R$  is the divider,
- $E$  is the trimmer, and
- $S$  is the chooser.

$R$  cuts the cake in the same way as he did above for the lone divider method giving the same valuations of slices as before.



i. In which of the following ways might  $E$  trim (given the valuations of the slices you found earlier). Circle all that apply.

- A. Remove  $\frac{1}{3}$  of the  $C_1$  part of  $S_1$ . no
- B. Remove  $\frac{1}{4}$  of the  $C_1$  part of  $S_1$ . no
- C. Remove  $\frac{1}{2}$  of the  $C_2$  part of  $S_2$ . no
- D. Remove all of the  $C_3$  part of  $S_2$ . no
- E. Remove  $\frac{1}{2}$  of the  $C_3$  part of  $S_3$ . no

Should trim  $S_3$  until it has value  $7/24$  to  $E$

$S_3 = \frac{1}{2} C_3 + 1 C_4$

F. Remove  $\frac{2}{3}$  of the  $C_4$  part of  $S_3$ .  $\rightarrow E \text{ val } 1/3 C_4 + 1/2 C_3 = 7/24$

ii. Suppose  $E$  decides to trim  $\frac{1}{3}$  of the  $C_4$  part plus all of the  $C_3$  part of  $S_3$ . The part of  $S_3$  left after the trimming is called  $S'_3$ .

$S_3' = S_3$

$-\frac{1}{3} C_4 - \frac{1}{2} C_3$

$= \frac{2}{3} C_4$

A. Fill in the blanks with the fraction of each component that occurs in each slice (this is just a volume measurement):

trimmings =  $\frac{0}{1} C_1 + \frac{0}{1} C_2 + \frac{1/2}{1} C_3 + \frac{1/3}{1} C_4$   
 $S'_3 = \frac{0}{1} C_1 + \frac{0}{1} C_2 + \frac{0}{1} C_3 + \frac{2/3}{1} C_4$

B. Fill in the valuations of each piece below (as fractions of the whole cake):

	$S_1$	$S_2$	$S'_3$	trimmings
$E$	$1/6$	$7/24$	$5/8$	$19/72$
$S$	$1/8$	$5/16$	$1/4$	$5/16$
$R$	$1/3$	$1/3$	$2/9$	$1/9$

C. The method tells us that first they must assign slices, ignoring the trimmings for now, this is Round 1. In what order will  $E, S$  and  $R$  choose slices in round 1?

- (1) S - chooses  $S_2$
- (2) E - must take  $S_3$
- (3) R - gets  $S_1$

iii. Now, they must divide up the trimmings. Suppose that at the end of round 1,  $S$  ends up with an untrimmed slice.

- Which of  $E, S$  and  $R$  should cut the trimmings?

$S$ , because they did not get the trimmed slice

- After the trimmings are cut, in what order to  $E, S$  and  $R$  choose a piece of the trimmings?

- (1)  $E$
- (2)  $R$
- (3)  $S$

- After both the slices and trimmings have been distributed, explain the following statement:

*R does not envy anyone else.*

End of round 1 -  $R$  has no envy, b/c  $R$  gets  $R$ 's fair share and other slices have value equal to or less than a fair share in  $R$ 's eyes.

Round 2 -  $R$  could only envy  $E$ . But  $E$  got trimmed slice, so even if  $E$  gets a slice that looks like total value of trimmings in  $R$ 's eyes,  $E$  will have  $\frac{2}{9} + \frac{1}{9} =$

$R$  val  $(S_3' + \text{trimmings}) =$  fair share to  $R$ ,

so  $R$  does not envy  $E$ .

