Homework 16: Chapter 16 Solutions

To 16.5 (Sols to extra exercises below)

16.5 Every equitable, pareto-optimal division is fair.

Proof. Suppose a division is unfair and it is equitable. Then every player is getting the same amount, which is less than their fair share. To objectively improve this, we could make the equal division - that way, everyone gets a slice that looks exactly the same, so everyone gets exactly their fair share which is an objective improvement.

Thus, a division which is fair and is not equitable is not pareto-optimal. □

To Ch 16 Day 2 (Sols to Day 1 see next pages)

1. A family is sharing a homogenous spread of Motzah, Latkes, Hard boiled eggs, and Parsley for passover. Their values for the components are given in "ideal cake" table below:

	Motzah	Latkes	HB Eggs	Parsley
A	1/4	0	3/4	0
B	2/3	1/3	0	0
C	1/6	1/2	1/3	0
D	1/4	1/4	1/4	1/4

(a) Is the equal division pareto-optimal in this example? Explain why or provide an objective improvement. The equal division is not pareto-optimal. An objective improvement is to give all the Parsley to D. Since A, B, and C do not want Parsley at all, they do not lose anything in this new division, and D benefits.

(b) Consider a cut which gives all the HB Eggs to A, all the Latkes to B, all the Motzah to C and all the Parsley to D. Is this division pareto-optimal? Explain why or provide an objective improvement. In tabular (table) form, this cut looks like

	S_A	S_B	S_C	S_D
Motzah	0	0	1	0
Latkes	0	1	0	0
HB Eggs	1	0	0	0
Parsley	0	0	0	1

Where S_A is A's slice, etc... Then the values players have for the different slices are

	S_A	S_B	S_C	S_D
A	3/4	0	1/4	0
B	0	1/3	2/3	0
C	1/3	1/2	1/6	0
D	1/4	1/4	1/4	1/4

This is clearly fair, but it isn't envy-free, and it is NOT pareto-optimal. In particular, if we just switch B's and C's slices, then they both get more without taking away from any other player. This is an objective improvement.

Challenge: Show this new division is pareto-optimal.

(c) If D gets everything, is this Pareto-optimal? Explain why or provide an objective improvement. Yes, this is pareto-optimal. If you give anything to another player, then you would have to take something away from D. So there are no objective improvements to this division.

From this example we can deduce that pareto-optimal does not imply fair and does not imply envy-free.

2. Prove: for $N = 5$ players, every envy-free division is fair. (Hint: show that if it is unfair to one player, then they have envy). Suppose there are 5 players and 5 slices to be allocated amongst them, and one player ends up not getting their fair share, which is $1/5$. If player A gets a slice with value less than $1/5$ to A, then adding up all A's values of the slices, we see that

$$\begin{aligned} 1 =& AvalS_A + AvalS_B + AvalS_C + AvalS_D + AvalS_E \\ \iff 1 <& 1/5 + AvalS_B + AvalS_C + AvalS_D + AvalS_E \\ \iff 4/5 <& AvalS_B + AvalS_C + AvalS_D + AvalS_E \end{aligned}$$

So there are four slices that A has different values for, and the sum of their values is $4 \times 1/5$. That means at least one slice must have value $\geq 1/5$ in A's eyes, which means A envies that players because A gets $< 1/5$.

To Ch 16 Day 1

1. Suppose avid pie-eaters A and B are dividing a homogeneous pie with pumpkin and Boston creme components.

   (a) If A likes pumpkin three times as much as Boston creme, describe A's ideal pie - ie, what proportions of the whole would A like each component to be?

   Solution. Let b be the amount of Boston creme A would like in A's ideal pie. Then A would like a $3b$ proportion of pumpkin. Solve:

$$\begin{aligned} \text{pump} \ + \ \text{BosCr} &= 1 \\ 3b + b &= 1 \\ b &= 1/4 \end{aligned}$$

   So A's ideal pie looks like $3b = 3/4$ pumpkin and $b = 1/4$ Boston creme.

   □

   (b) If B likes Boston creme four times as much as pumpkin, describe B's ideal pie. (you might want to put this information in a table) Let p be the amount of pumpkin B would like in B's ideal pie. Solve:

$$p + 4p = 1 \iff p = 1/5$$

   to determine that B's ideal pie is 1/5 pumpkin and 4/5 Boston creme.
   The ideal pie table for A and B is:

	Pump	BosCr
A	3/4	1/4
B	1/5	4/5

   (c) Consider these two cuts:

	S_1	S_2
Pump	1/3	2/3
BosCr	1	0

	T_1	T_2
Pump	5/6	1/6
BosCr	0	1

   Remember: the cut on the left is 1/3 pumpkin and all the Boston creme in the first slice, S_1, and the rest in S_2. For the cut on the right, T_1 has 5/6 pumpkin and none of the Boston creme.

   Which cut(s) would A make to guarantee getting A's fair share? The value of S_1 to A is

$$\frac{3}{4} \times \frac{1}{3} + \frac{1}{4} \times 1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

   So S_2 must also have value 1/2 to A, and A could make this cut and guarantee getting A's fair share.

The value of T_1 to A is

$$\frac{3}{4} \times \frac{5}{6} + \frac{1}{4} \times 0 = \frac{5}{4 \times 2} = \frac{5}{8}$$

which looks good to A, but the value of T_2 to A is then only $\frac{3}{8}$, so A cannot guarantee getting A's fair share if A makes this cut.

(d) If A makes a good cut for playing I cut you choose using one of the cuts above, which slice would B pick? A would make the S_1, S_2 cut to guanrantee getting their fair share in I cut you choose. Then B's value of S_1 is very high, because B gets 4/5 of value out of the whole Boston Creme component which is in S_1. So B would pick S_1, which is clearly more valuable to B (an exact compuation will verify this).

(e) There are many cuts that A could make to guarantee getting A's fair share in I cut you choose. Can you find others? If we put p amounts of pumpkin and b amounts of Boston creme in S_1, then to have value 1/2 in A's eyes, these proportions must satisfy the equation

$$\frac{3}{4}p + \frac{1}{4}b = \frac{1}{2}$$

Choose any values for p and you can solve for b - you have a value division as long as both p and b are between 0 and 1. Here are a few examples:

- $p = 1/2, b = 1/2$
- $p = 2/3, b = 0$
- $p = 5/12, b = 3/4$