Homework 14: Chapter 14 Solutions

Other Exercises:

1. Suppose there are three bidders with bids a, b, c. Prove: If the winning bidder A is an average bidder, then $q = \frac{1}{3}$.

   Solution. If A is an average bidder then $a = \frac{a+b+c}{3}$. By our work in class, we know

   $$q = \frac{w}{S} = a \times \frac{1}{a+b+c} = \frac{\cancel{a+b+c}}{3} \times \frac{1}{\cancel{a+b+c}} = \frac{1}{3}$$

   We can conclude that the equitable arrangement is fair, because every bidder is getting exactly their fair share:

   $$x_A = aq = \frac{a}{3}$$
   $$x_B = bq = \frac{b}{3}$$
   $$x_C = cq = \frac{c}{3}$$

   □

2. Is every equitable compensation arrangement envy-free? (Check some examples to help formulate your answer).

   Solution. No, for example: Suppose $N = 3$, a is an average bidder, and $b < c$. Then $x_B = bq < cq = x_C$ so there is envy immediately! (Try plugging in numbers for a, b, and c to make this statement clearer.) □

3. If the answer to Question 2 is yes: prove it. If the answer to Question 2 is no: Can you find a (nontrivial) equitable compensation arrangement which *is* envy-free?

   Sure can. We could've fixed the last example by making everyone's bids equal. Then A is still an average bidder, and by Question 1 the equitable arrangement is that everyone gets $\frac{a}{3}$ (since all bids are equal).

   But that isn't particularly exciting.

   Another tricky thing: if there are two bidders, make the highest bidder a winning bidder and find the equitable arrangement. The highest bidder is at least average so by Prop 14.9, this is fair, and by Corollary 13.15 the equitable compensation arrangement will be envy-free.

   But this is also not particularly exciting.

In general, to be envy free, we need (at a minimum) that all payouts are equal. To be equitable, we need

$$\begin{aligned} x_A &= aq \\ x_B &= bq \\ x_C &= cq \\ &\vdots \end{aligned}$$

Then to be envy-free AND equitable, (and supposing A is a winning bidder because why not), we need

$$x_B = x_C = x_D = \cdots \iff bq = cq = dq = \cdots \iff b = c = d = \cdots$$

In other words: being equitable and envy-free is only possible when all the (non-winning) bids are equal. We still don't know for sure if this compensation arrangement is envy-free! It also needs to be fair! So we need $a \geq m = \frac{S}{N}$. This condition is made much more specific in this context because we've required all other bids to be equal. So in particular,

$$\begin{aligned} a \geq m &\iff a \geq \frac{S}{N} \\ &\iff a \geq \frac{a+b+c+d+\cdots}{N} \\ &\iff a \geq \frac{a+(N-1)b}{N} \\ &\iff Na \geq a+(N-1)b \\ &\iff (N-1)a \geq (N-1)b \\ &\iff a \geq b \end{aligned}$$

We just proved that not only must a be at least average, but if every other bid is equal this is equivalent to saying a is bigger than all those bids: i.e., A is a highest bidder. We actually just proved:

Baby Proposition. The equitable compensation arrangement is envy-free if and only if the winning bidder is a highest bidder and all other bids are equal. Otherwise, it is impossible to find a compensation arrangement which is both equitable and envy-free.