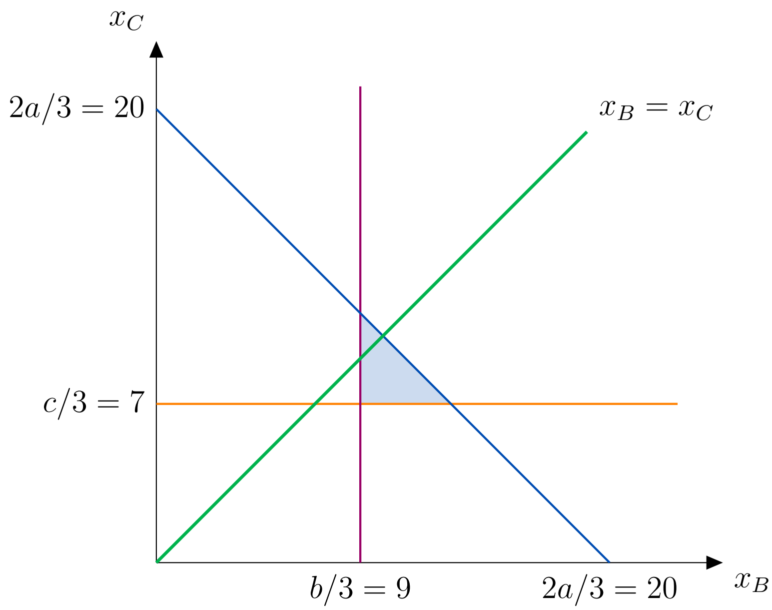
Homework 12: Chapter 13 Solutions

1. In class, we made the fairness triangle for the example with bids $a = 30, b = 27, c = 21$ and A the winning bidder. The fairness triangle in the (x_B, x_C)-plane is shaded in blue in the picture below. The green line represents all compensation arrangements with equal compensation amounts.



(a) **Remark**. The vertical pink line $x_B = 9$ represents all compensation arrangements such that B gets exactly B's fair share, which is 9. What does the horizontal orange line represent? C getting exactly C's fair share

(b) The slanted blue line is all the compensation arrangements with $x_B + x_C = 2a/3$. **Explain**: on this line, A is getting *exactly* A's fair share. Off this line, A is getting either more or less than A's fair share.
On this line $x_B + x_C = 2a/3$, we can find A's payout:

$$x_A = a - x_B - x_C = a - (x_B + x_C) = a - \frac{2a}{3} = \frac{3a - 2a}{3} = a/3$$

If $x_B + x_C < 2a/3$, then $-(x_B + x_C) > 2a/3$, so

$$x_A = a - x_B - x_C = a - (x_B + x_C) > a - \frac{2a}{3} = a/3$$

A gets more than A's fair share.

If $x_B + x_C > 2a/3$, then the inequality above is flipped and A spent too much money to get A's fair share out of the object that A won.

(c) Plot the compensation arrangement

$$x_B = 11 \qquad\qquad x_C = 9$$

Is this compensation arrangement going to be fair? Envy-free? Yes, it will be fair, but it will not be envy-free. Point is plotted in black.

(d) What points are associated to the three corners of the fairness triangle? Top left: $(9, 11)$. Bottom left: (9,7). Bottom right: (13,7).

(e) Find the intersection point between $x_B = x_C$ and the line where A gets A's fair share. Plot it in the plane. Plotted in blue:

$$x_A = a/3 \iff x_B + x_C = \frac{2a}{3}$$

Intersected with $x_B = x_C$:

$$x_B + x_B = \frac{2a}{3} \iff 2x_B = \frac{2a}{3} \iff x_B = x_C = \frac{a}{3}$$

The intersection point is $(a/3, a/3) = (10, 10)$. (btw that was a general proof)

2. In class we also studied the situation when B was the winning bidder. Consider the same bids $a = 30, b = 27, c = 21$, and assume B is a winning bidder. *Paychecks are now going to A and C.*

(a) In the (x_A, x_C)-plane, graph all compensation arrangements fair to A. right of vertical pink in Figure 1

(b) In the (x_A, x_C)-plane, graph all compensation arrangements fair to B. below blue in Figure 1

(c) In the (x_A, x_C)-plane, graph all compensation arrangements fair to C. above orange in Figure 1

(d) Draw the fairness triangle. Find the coordinates for the three corners of the triangle. (10,7), (10,8), (11,7)

(e) Find the intersection point between $x_A = x_C$ and the line where B gets B's fair share. Plot it in the plane. (9,9)

(f) Is the compensation arrangement associated to that point fair to

A? no

B? yes

C? yes

(g) Draw the line $x_A = x_C$. Draw a line from the origin to (9,9)

(h) Can you find an envy-free compensation arrangement for this example? If yes, give one. No, because the line $x_A = x_C$ representing equal compensation amounts does not intersect the fairness triangle

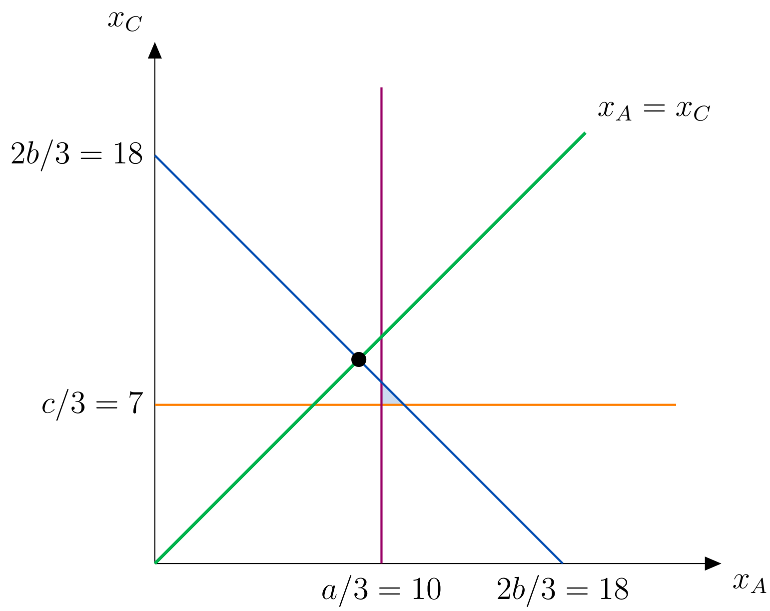


Figure 1

3. Consider again three bidders who submit bids $a = 15, b = 9, c = 21$.

   (a) What are the fair shares? A: 5, B: 3, C: 7

   (b) What is the average bid? 15

   (c) Suppose from here on out that A is the winning bidder. Plot the region of the plane where B and C are getting their fair shares. B: to the right of the pink line in Figure 2. C: above the orange line in Figure 2.

   (d) What is the intersection point where B and C get *exactly* their fair shares? Call this intersection point P. $P = (3, 7)$

   (e) **Prove**: P is on the line which represents A getting *exactly* A's fair share. We want to show P is on the line $x_B + x_C = 10$. This is true because:

   $$3 + 7 = 10$$

   So we conclude that the line $x_B + x_C = 10$ crosses through P.

   (f) Draw the "fairness triangle". It is exactly the point P. No other points are in the fairness triangle.

   (g) Draw the line $x_B = x_C$. Using the strategy outlined in the last problem: where does the line $x_B = x_C$ hit the line representing A getting A's fair share? It will be at $(a/3, a/3) = (5, 5)$. Use this to plot the line and see where it sits on your graph (see dashed lines in Figure 2).

   (h) Do there exist envy-free compensation arrangements for this example? No, because the line representing equal compensation amounts does not cross the fairness triangle.

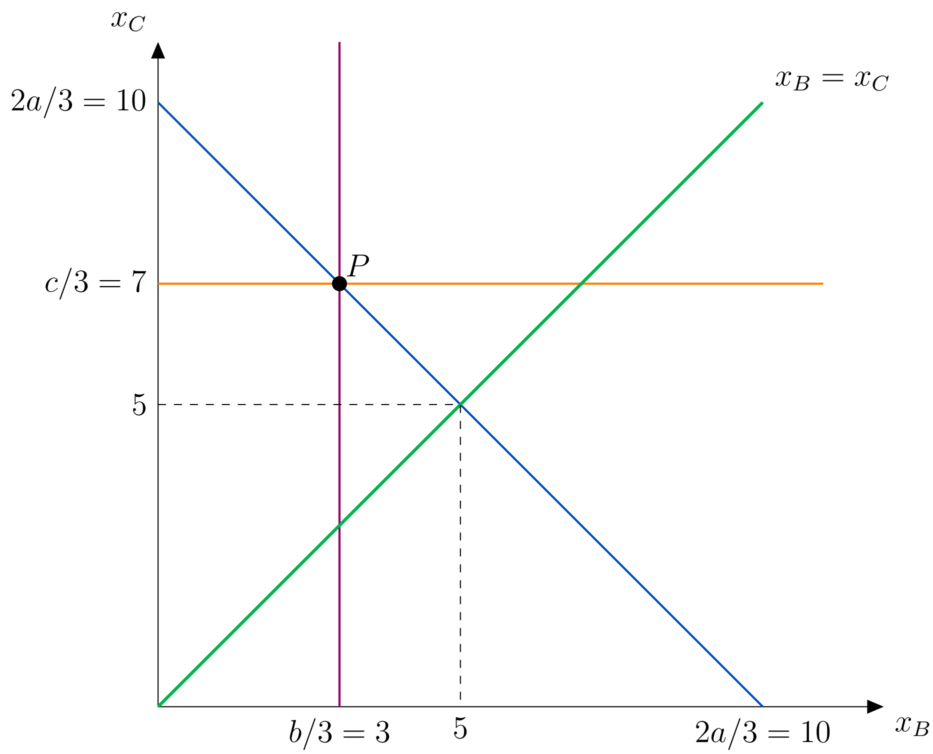


Figure 2

4. (Optional) Prove: For any number of bidders and any bids, the sum of the fair shares is equal to the average bid.

   Proof. Suppose the n bids are $b_1, b_2, \ldots, b_n$. Then the fair shares are $b_1/n, b_2/n, \ldots, b_n/n$. If we add them all up:

$$\frac{b_1}{n} + \frac{b_2}{n} + \cdots + \frac{b_n}{n} = \frac{b_1 + b_2 + \cdots + b_n}{n} = \frac{S}{n} = m$$

$\square$